# Gauge invariant perturbations of black holes using the modified Newman-Penrose formalism

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**Abstract**

We present the results of recent work on a gauge invariant approach to the gravitational perturbations of the Kerr space-time, using the modified Newman-Penrose formalism. The techniques used are a generalisation of those developed in the Schwarzschild case. The perturbed Bianchi identities are written in a form involving only certain tetrad and coordinate-gauge invariant field quantities. The integrability conditions for the perturbed Bianchi identities then provide a system of gauge invariant wave-like gravitational perturbation equations, and the transformations which relate them to one another. The analysis is coordinate-free, and provides a geometric and gauge invariant explanation of the transformations relating the Teukolsky equation and a Regge-Wheeler-like equation in the Kerr background. The electromagnetic and gravitational perturbations of the Reissner-Nordstrom space-time can also be investigated using this approach. We present a gauge invariant coupled wave-like electromagnetic-gravitational perturbation equation in this case, which may be decoupled.

## 1. Introduction

The perturbations of the Kerr-Newman black hole have defied many attempts at clarification in the past (see for example Chandrasekhar [1], Fackerell [2]). In part this is due to the complicated nature of the coupling between electromagnetic and gravitational fields in electrovac space-times, and to the presence of angular momentum in the background. An understanding of the perturbations of the Kerr-Newman black hole is important since the Kerr-Newman black hole is the most general stationary black hole solution of the Einstein equations. Although there is widespread belief that Kerr-Newman black holes are stable to linear perturbations, there is no proof that this is the case.

Recently Fernandes and Lun [3] have developed a successful gauge invariant technique for investigating the gravitational perturbations of the Schwarzschild black hole, using the modified Newman-Penrose formalism. An extension of this approach to the perturbations of the Kerr and Reissner-Nordström black holes is an important precursor to the study of the Kerr-Newman space-time. In the Kerr case, the effects of angular momentum on the perturbation problem can be investigated without the added complication of a background electric charge. In the Reissner-Nordström case, the effects of charge can be investigated in the absence of angular momentum. Once a full gauge invariant analysis of the Kerr and Reissner-Nordström black holes is complete, we will be in a strong position to attack the perturbations of the Kerr-Newman space-time.

The basis for the gauge invariant formalism developed in [3] is that the perturbed Bianchi identities, like the perturbed Maxwell equations, are gauge invariant:

$$\pounds_u\left(R^a_{\ b[cd;e]}\right) = 0$$

where $R^a_{\ b[cd;e]}$ refers to the background space-time, and $u^a$ is an arbitrary vector field (refer to Lun [4], Stewart and Walker [5]). However, in contrast to the Maxwell scalars in vacuum backgrounds, the usual Newman-Penrose field quantities in the perturbed Bianchi identities in Schwarzschild space-time are not gauge invariant (with the exception of $\Psi_{4B}$ and $\Psi_{0B}$). Since they are gauge invariant, the perturbed Bianchi identities allow the definition of a set of new gauge invariant field quantities, which are boost- and spin-weighted quantities related to the Newman-Penrose Weyl scalars and spin coefficients. The significance of these field quantities is that they can be expressed as gauge invariant combinations of the relevant perturbed metric components. The Bianchi identities are then rewritten solely in terms of these new gauge invariant field quantities.

When written in this form, we view the Bianchi identities as the gravitational analogues of the Maxwell equations. Thus, modelling our approach on the electromagnetic perturbations of the Schwarzschild black hole, we derive a system of gauge invariant gravitational wave equations, and the transformations relating them to one another, from the integrability conditions for the perturbed Bianchi identities. The wave equations are the spin-weight $\pm 2$ Bardeen-Press [6] equations, two spin-weight 0 (gauge invariant) Regge-Wheeler [7] equations, and two new spin-weight $\pm 1$ gravitational wave equations. The transformations between the equations require some higher-order commutators, which can be derived from the Newman-Penrose commutation relations. Although the Zerilli [8] equation does not arise naturally in this approach, it can be constructed in a gauge invariant manner. The transformations between the Bardeen-Press and Regge-Wheeler equations derived by Chandrasekhar [1] and Sasaki and Nakamura [9, 10] arise in this analysis in a geometric and gauge invariant manner. To a large extent, the success of these techniques is due to the geometry of the background space-time.

The purpose of the present article is to present results of recent work on the generalisation of the approach in [3] to the Kerr and Reissner-Nordström space-times. More extensive derivations and explanation of these results are given in [11], [12] and [13].

The approach to perturbations developed in [3] is heavily dependent upon expansion in spin-weighted spherical harmonics, which in turn requires the introduction of coordinates. In the Kerr case, expansion into tensorial spheroidal harmonics is not possible in general. However our techniques do generalize in a natural way, with the advantage that we are not required to make any use of coordinates at all in our perturbation analysis. Thus, the approach which we used in the Schwarzschild case can be extended to treat the electromagnetic and gravitational perturbations of the Kerr space-time.

After identifying the appropriate gauge invariant field quantities in the Kerr case, the perturbed Bianchi identities may be cast into a form involving only these quantities. The perturbed Bianchi identities then give rise to gauge invariant wave-like perturbation equations for our field quantities, and the transformations which relate one to another. The equations are the well known spin-weight $-2$ Teukolsky [14] equation, the (gauge invariant) Kerr analogue of the Regge-Wheeler equation, and a new gauge invariant gravitational perturbation equation. The transformation between the Teukolsky equation and the analogue of the Regge-Wheeler equation arises in this way.

The extension of the gauge invariant formalism to the Kerr space-time is a technical one. The presence of angular momentum in the Kerr background does not pose as great a conceptual problem as the existence of electromagnetic charge in the Reissner-Nordström space-time. In contrast to the vacuum case, the perturbed Maxwell scalars, as well as most of the perturbed Weyl scalars, are not gauge invariant in the Reissner-Nordström background. Coupling between gravitational and electromagnetic fields complicates the perturbation problem greatly. Furthermore, the structure of the Newman-Penrose equations changes somewhat. Both electromagnetic and gravitational field quantities are present in the perturbed Maxwell equations and Bianchi identities, and the perturbed Bianchi identities involve a mixture of quantities of the opposite spin-weight. Nevertheless, the gauge invariant technique can be extended to this case as well.

After identifying the gauge invariant field quantities in the Reissner-Nordström case, the perturbed Bianchi identities and Maxwell equations may be cast into a form involving only these quantities [13]. The integrability conditions provide a system of gauge invariant coupled electromagnetic-gravitational perturbation equations for the Reissner-Nordström space-time. In the spin-weight 0 case, we have a coupled Regge-Wheeler-like equation for the gauge invariant quantities $Im(\Phi_{1B})$ and $Im(\Psi_{2B})$. This equation may be decoupled, providing a pair of gauge invariant Regge-Wheeler-type equations which agree precisely with Moncrief's [15] equations for the "odd-parity" perturbations of the Reissner-Nordström black hole, derived using the Hamiltonian formulation. Therefore, this work establishes a link between the modified Newman-Penrose approach and the Hamiltonian approach to perturbations. The results may be derived directly from the Newman-Penrose equations in a naive fashion, without extending the gauge invariant formalism. In this article we present the results of this analysis, which are a precursor to the full gauge invariant analysis of the perturbations of Reissner-Nordström spacetime.

Naturally, all of the results presented here reduce to their Schwarzschild analogues when the angular momentum or charge vanishes. We adopt the usual convention of denoting perturbed quantities by a subscript $B$. Background quantities have no subscript.

## 2. Kerr Space-time

Describing Kerr space-time in the Newman-Penrose formalism, we have:

$$\kappa = \sigma = \lambda = \nu = \Phi_0 = \Phi_1 = \Phi_2 = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0,$$

$$þ\Psi_2 = 3\varrho\Psi_2, \quad þ'\Psi_2 = -3\mu\Psi_2, \quad ð\Psi_2 = 3\tau\Psi_2, \quad ð'\Psi_2 = -3\pi\Psi_2, \tag{1}$$

and[1]

$$\frac{\varrho}{\overline{\varrho}} = \frac{\mu}{\overline{\mu}} = \frac{\pi}{\overline{\tau}} = \frac{\tau}{\overline{\pi}}. \tag{2}$$

Furthermore, from the Ricci identities, equations (1), (2), and the Newman-Penrose commutators, we derive a more complete set of relationships between the derivatives of the spin-coefficients in Kerr background:

$$\begin{gathered}
\text{þ}\varrho = \varrho^2,\ \eth\varrho = (\varrho - \overline{\varrho})\tau,\ \eth'\varrho = -\varrho(\overline{\tau} + \pi) - \pi(\varrho - \overline{\varrho}) \\
\text{þ}'\mu = -\mu^2,\ \eth'\mu = -(\mu - \overline{\mu})\pi,\ \eth\mu = \mu(\tau + \overline{\pi}) + \tau(\mu - \overline{\mu}) \\
\text{þ}\tau = \varrho(\tau + \overline{\pi}),\ \text{þ}'\tau = -\eth\mu,\ \eth\tau = \tau^2 \\
\text{þ}\pi = -\eth'\varrho,\ \text{þ}'\pi = -\mu(\overline{\tau} + \pi),\ \eth'\pi = -\pi^2 \qquad (3) \\
\text{þ}'\varrho - \eth'\tau = -\varrho\overline{\mu} - \tau\overline{\tau} - \Psi_2,\ \text{þ}\mu - \eth\pi = \overline{\varrho}\mu + \pi\overline{\pi} + \Psi_2, \\
\text{þ}\mu + \text{þ}'\varrho = \eth\pi + \eth'\tau = 0, \qquad (4)
\end{gathered}$$

and

$$\text{þ}\mu = \varrho\mu + \pi(\overline{\pi} + \tau) + \frac{1}{2}\Psi_2 + \frac{\varrho}{2\overline{\varrho}}\overline{\Psi}_2,$$

where $\varrho \neq 0,\ \pi \neq 0$. In the Schwarzschild limit, $\dfrac{\varrho}{\overline{\varrho}} = 1$.

Using this information, together with the Newman-Penrose commutators, we derive the following commutation relation for general $(p,q)$ weighted quantities

$$\begin{aligned}
&\Big(\text{þ} - \Big(a + \frac{p}{2}\Big)\varrho - \Big(b + 1 + \frac{q}{2}\Big)\overline{\varrho}\Big)\Big(\eth - \Big(a + \frac{p}{2}\Big)\tau + \Big(b - \frac{q}{2}\Big)\overline{\pi}\Big) \\
&\quad - \Big(\eth - \Big(a + \frac{p}{2}\Big)\tau + \Big(b + 1 - \frac{q}{2}\Big)\overline{\pi}\Big)\Big(\text{þ} - \Big(a + \frac{p}{2}\Big)\varrho - \Big(b + \frac{q}{2}\Big)\overline{\varrho}\Big) = q\eth\overline{\varrho} \qquad (5)
\end{aligned}$$

where $a$ and $b$ are arbitrary constants. The other commutators can be obtained from (5) by complex conjugation and applying the GHP [16] prime. Recalling that $(p,q)$ becomes $(q,p)$ under conjugation and $(p,q)$ becomes $(-p,-q)$ under the prime, these identities simplify our computations greatly.

### *2.1 The Perturbed Bianchi Identities With Gauge Invariant Fields*

In Kerr background, the perturbed Bianchi identities are:[2]

$$\begin{aligned}
(\text{þ} - \varrho)\Psi_{4B} &= (\eth' + 4\pi)\Psi_{3B} - 3\lambda_B\Psi_2 && (6)\\
(\text{þ} - 2\varrho)\Psi_{3B} &= (\eth' + 3\pi)\Psi_{2B} + (\eth' + 3\pi)_B\Psi_2 && (7)\\
(\eth - \tau)\Psi_{4B} &= (\text{þ}' + 4\mu)\Psi_{3B} - 3\nu_B\Psi_2 && (8)\\
(\eth - 2\tau)\Psi_{3B} &= (\text{þ}' + 3\mu)\Psi_{2B} + (\text{þ}' + 3\mu)_B\Psi_2. && (9)
\end{aligned}$$

[1] Chandrasekhar [1], p324.
[2] Refer to Penrose and Rindler [17], Eqs. (4.12.36)–(4.12.39) and Eqs. (4.12.32).

Only half of the Bianchi identities need be considered, since the others may be obtained by applying the prime operator. The following perturbed Ricci identities will also be used:

$$\begin{aligned}(þ' + \mu + \overline{\mu})\lambda_B - (ð' + \pi - \overline{\tau})\nu_B &= -\Psi_{4B} &(10)\\ ð\lambda_B - (ð' + \pi)_B\mu - (ð' + \pi)\mu_B &= -\overline{\mu}\pi_B - \overline{\mu}_B\pi + (\varrho - \overline{\varrho})\nu_B - \Psi_{3B} &(11)\\ (þ' + \mu)_B\pi + (þ' + \mu)\pi_B - þ\nu_B &= -\overline{\tau}_B\mu - \overline{\tau}\mu_B - (\tau + \overline{\pi})\lambda_B - \Psi_{3B} &(12)\end{aligned}$$

To put the Bianchi identities into a form which, like the Maxwell equations, involves only gauge invariant field quantities, we define:

$$\begin{aligned}\widetilde{\Psi}_{3B} &:= (ð' + 4\pi)\Psi_{3B} - 3\lambda_B\Psi_2\\ \widetilde{\Psi}_{2B} &:= (ð' + 5\pi)\Big[(ð' + 3\pi)\Psi_{2B} + (ð' + 3\pi)_B\Psi_2\Big] - 2\Psi_{3B}(ð'\varrho) - 3\Psi_2(þ\lambda_B). \qquad (13)\end{aligned}$$

These are gauge invariant weighted quantities of $(p,q)$ type $(-3,1)$ and $(-2,2)$, respectively. Each field quantity has spin-weight $-2$. The correspondence between these field quantities and the quantities used in the Schwarzschild case[3] are:

$$\begin{aligned}\widetilde{\Psi}_{3B} &\longleftrightarrow ð'\widehat{\Psi}_{3B}\\ \widetilde{\Psi}_{2B} &\longleftrightarrow ð'ð'\widehat{\Psi}_{2B}.\end{aligned}$$

The origin of the quantities (13) is clear. Firstly, $\widetilde{\Psi}_{3B}$ is defined as the right hand side of the Bianchi identity (6). The field quantity $\widetilde{\Psi}_{2B}$ is then found from (7), by operating with $(ð' + 5\pi)$ and using the complex conjugate of the commutator (5), with $(p,q) = (-2,0)$ and $(a,b) = (0,3)$.[4] The left hand side becomes

$$\begin{aligned}(ð' + 5\pi)(þ - 2\varrho)\Psi_{3B} &= (þ - 3\varrho)(ð' + 4\pi)\Psi_{3B} + 2\Psi_{3B}(ð'\varrho)\\ &= (þ - 3\varrho)(\widetilde{\Psi}_{3B} + 3\Psi_2\lambda_B) + 2\Psi_{3B}(ð'\varrho).\end{aligned}$$

Simplifying this, using equation (1), we get

$$(þ - 3\varrho)\widetilde{\Psi}_{3B} = (ð' + 5\pi)\Big[(ð' + 3\pi)\Psi_{2B} + (ð' + 3\pi)_B\Psi_2\Big] - 2\Psi_{3B}(ð'\varrho) - 3\Psi_2(þ\lambda_B)$$

and the right hand side defines $\widetilde{\Psi}_{2B}$. Thus, in particular, each of (13) is gauge invariant, and this is discussed below.

Now the Bianchi identities may be expressed solely in terms of the gauge invariant quantities (13):

$$\begin{aligned}(þ - \varrho)\Psi_{4B} &= \widetilde{\Psi}_{3B} &(14)\\ (þ - 3\varrho)\widetilde{\Psi}_{3B} &= \widetilde{\Psi}_{2B} &(15)\\ \Big[(ð' + 4\pi - \overline{\tau})(ð - \tau) + 3\Psi_2\Big]\Psi_{4B} &= (þ' + 4\mu + \overline{\mu})\widetilde{\Psi}_{3B} &(16)\\ (þ' + 4\mu + \overline{\mu})\widetilde{\Psi}_{2B} &= \Big[(ð' + 4\pi - 2\overline{\tau})(ð - 2\tau) + ((ð' + \overline{\tau})\tau - (þ + \overline{\varrho})\mu)\Big]\widetilde{\Psi}_{3B}\\ &\quad - (\varrho - \overline{\varrho})(þ' + 4\mu + \overline{\mu})\widetilde{\Psi}_{3B}\\ &\quad + 3\Big[(2\varrho - \overline{\varrho})\Psi_2 - (ð'\varrho)(ð - \tau)\Big]\Psi_{4B}. &(17)\end{aligned}$$

[3] See [3] Eq. (3.29).

[4] That is, write down the conjugate of (5), noting that $p \to q$ and $q \to p$. Then substitute the given values of $(p,q)$ into this new commutator. The same will apply below.

To show that the identities (16) and (17) are identical with (8) and (9), respectively, requires some work. For example, acting on (8) with $(ð' + 4\pi - \overline{\tau})$, using the primed version of the commutator (5) $((p,q) = (-2,0),\ (a,b) = (3,0))$ and the Ricci identity (10) we have

$$(ð' + 4\pi - \overline{\tau})(ð - \tau)\Psi_{4B} = (þ' + 4\mu + \overline{\mu})\Big((ð' + 4\pi)\Psi_{3B} - 3\lambda_B\Psi_2\Big) - 3\Psi_2\Psi_{4B}.$$

So (8) and (16) are identical. A similar calculation reveals that (8) and (17) are identical, and the details are given in [11].

### *2.2 Proof Of Gauge Invariance*

Since each of (13) can be related to the gauge invariant Newman-Penrose field $\Psi_{4B}$ in the perturbed Bianchi identities above, it is evident that they are gauge invariant. Alternatively, we may check that the field quantities are indeed gauge invariant in a brief calculation. A combined infinitesimal coordinate-gauge transformation and infinitesimal Lorentz transformation has the following effect (see Lun [4]):

$$\begin{array}{rcl} \Psi_{4B} & \longmapsto & \Psi_{4B} \\ \Psi_{3B} & \longmapsto & \Psi_{3B} + 3v\Psi_2 \\ \lambda_B & \longmapsto & \lambda_B + (ð' + \pi)v. \end{array} \qquad (18)$$

Thus

$$\begin{array}{rcl} \widetilde{\Psi}_{3B} & := & (ð' + 4\pi)\Psi_{3B} - 3\lambda_B\Psi_2 \\ & & \longmapsto (ð' + 4\pi)\Psi_{3B} + 3\Psi_2(ð' + \pi)v - 3\lambda_B\Psi_2 - 3\Psi_2(ð' + \pi)v = \widetilde{\Psi}_{3B}. \end{array}$$

On the other hand, using equation (7), $\widetilde{\Psi}_{2B}$ may be written

$$\widetilde{\Psi}_{2B} = (ð' + 5\pi)(þ - 2\varrho)\Psi_{3B} - 2\Psi_{3B}(ð'\varrho) - 3\Psi_2(þ\lambda_B),$$

and, under the gauge transformations, $\widetilde{\Psi}_{2B}$ becomes

$$\widetilde{\Psi}_{2B} \longmapsto \widetilde{\Psi}_{2B} + 3\Psi_2[(ð' + 2\pi)(þ + \varrho) - þ(ð' + \pi) - 2(ð'\varrho)]v.$$

The terms involving $v$ on the right hand side vanish identically, using the complex conjugate of the commutator (5), with $(p,q) = (-2,0),\ (a,b) = (0,0)$.

### *2.3 Wave-like Gravitational Perturbation Equations*

The perturbed Bianchi identities (14)–(17) then allow us to derive a system of wave-like gravitational perturbation equations in the following way. Acting on (14) with $(þ' + 4\mu + \overline{\mu})$ and using (16) we have

$$\Big[(þ' + 4\mu + \overline{\mu})(þ - \varrho) - (ð' + 4\pi - \overline{\tau})(ð - \tau) - 3\Psi_2\Big]\Psi_{4B} = 0. \qquad (19)$$

This is the usual spin-weight $-2$ Teukolsky equation.

Operating on (16) with $(\text{þ} - 2\varrho - \overline{\varrho})$ we get

$$(\text{þ} - 2\varrho - \overline{\varrho})(\text{þ}' + 4\mu + \overline{\mu})\widetilde{\Psi}_{3B} = (\text{þ} - 2\varrho - \overline{\varrho})\Big[(\eth' + 4\pi - \overline{\tau})(\eth - \tau) + 3\Psi_2\Big]\Psi_{4B}. \qquad (20)$$

Now, using the complex conjugate of (5) (with $(p,q) = (-3,-1)$, $(a,b) = (\frac{3}{2}, \frac{5}{2})$), and (5) (with $(p,q) = (-4,0)$, $(a,b) = (3,0)$), we derive the following commutator for weighted quantities of $(p,q)$ weight $(-4,0)$:

$$(\text{þ} - 2\varrho - \overline{\varrho})(\eth' + 4\pi - \overline{\tau})(\eth - \tau) = (\eth' + 5\pi - \overline{\tau})(\eth - \tau + \overline{\pi})(\text{þ} - \varrho) - 3(\eth'\varrho)(\eth - \tau).$$

Hence, using (1), equation (20) becomes

$$\begin{aligned}&(\text{þ} - 2\varrho - \overline{\varrho})(\text{þ}' + 4\mu + \overline{\mu})\widetilde{\Psi}_{3B}\\ &\quad = (\eth' + 5\pi - \overline{\tau})(\eth - \tau + \overline{\pi})(\text{þ} - \varrho)\Psi_{4B} + 3\Psi_2(\text{þ} + \varrho - \overline{\varrho})\Psi_{4B} - 3(\eth'\varrho)(\eth - \tau)\Psi_{4B}.\end{aligned}$$

From (14) this may be rewritten as a wave-like perturbation equation for $\widetilde{\Psi}_{3B}$

$$\begin{aligned}&\Big[(\text{þ} - 2\varrho - \overline{\varrho})(\text{þ}' + 4\mu + \overline{\mu}) - (\eth' + 5\pi - \overline{\tau})(\eth - \tau + \overline{\pi}) - 3\Psi_2\Big]\widetilde{\Psi}_{3B}\\ &\qquad = 3\Big[\Psi_2(2\varrho - \overline{\varrho}) - (\eth'\varrho)(\eth - \tau)\Big]\Psi_{4B}. \qquad (21)\end{aligned}$$

In a similar fashion, from (15) and (17) we derive the following perturbation equation for $\widetilde{\Psi}_{2B}$:

$$\begin{aligned}&\Big[(\text{þ} - 3\varrho - 2\overline{\varrho})(\text{þ}' + 4\mu + \overline{\mu}) - (\eth' + 5\pi - 2\overline{\tau})(\eth - 2\tau + \overline{\pi}) + ((\text{þ} + \overline{\varrho})\mu - (\eth' + \overline{\tau})\tau)\Big]\widetilde{\Psi}_{2B}\\ &\quad = 6\varrho\Psi_2(\varrho - \overline{\varrho})\Psi_{4B} + 3\varrho\Psi_2\widetilde{\Psi}_{3B} - 4(\eth'\varrho)(\eth - \tau + \overline{\pi})\widetilde{\Psi}_{3B} + 2(\tau + \overline{\pi})(\eth'\varrho)\widetilde{\Psi}_{3B}\\ &\qquad + \widetilde{\Psi}_{3B}(\text{þ} - 2\overline{\varrho})\Big[(\eth' + \overline{\tau})\tau - (\text{þ} + \overline{\varrho})\mu\Big] \qquad (22)\end{aligned}$$

Using (1) and (4), the right hand side simplifies to

$$6\varrho\Psi_2(\varrho - \overline{\varrho})\Psi_{4B} - 2(\eth'\varrho)(2\eth - 3\tau + \overline{\pi})\widetilde{\Psi}_{3B} + 2\widetilde{\Psi}_{3B}\Big[(\text{þ} - 2\overline{\varrho})\eth'\tau - \overline{\varrho}(\eth + \overline{\pi})\pi\Big],$$

which collapses if $\pi = 0$, $\Psi_2 = \overline{\Psi}_2$ (which implies $\varrho = \overline{\varrho}$), and we recover precisely the Schwarzschild result. Importantly, equations (21) and (22) are the Kerr analogues of the spin-weight $-1$ and spin-weight 0 (Regge-Wheeler) equations for the perturbations of the Schwarzschild space-time, respectively.

Equivalent forms of equation (21) are:

$$\begin{aligned}&\Big[(\text{þ}' + 4\mu + \overline{\mu})(\text{þ} - 3\varrho) - (\eth' + 4\pi - 2\overline{\tau})(\eth - 2\tau) + ((\text{þ} + \overline{\varrho})\mu - (\eth' + \overline{\tau})\tau)\Big]\widetilde{\Psi}_{3B}\\ &\qquad = 3\Big[(2\varrho - \overline{\varrho})\Psi_2 - (\eth'\varrho)(\eth - \tau)\Big]\Psi_{4B} - (\varrho - \overline{\varrho})(\text{þ}' + 4\mu + \overline{\mu})\widetilde{\Psi}_{3B} \qquad (23)\end{aligned}$$

or

$$\Big[(þ' + 4\mu + \overline{\mu})(þ - 2\varrho - \overline{\varrho}) - (ð' + 4\pi - 2\overline{\tau})(ð - 2\tau) + 2\Psi_2 - \overline{\Psi}_2 + 2\overline{\varrho}\mu + 3(ð\pi) - (ð'\overline{\pi})\Big]\widetilde{\Psi}_{3B}$$
$$= 3\Big[(2\varrho - \overline{\varrho})\Psi_2 - (ð'\varrho)(ð - \tau)\Big]\Psi_{4B}. \tag{24}$$

These can be derived from (15) and (17) or alternatively from (21), using the Newman-Penrose commutators $[þ, þ']$ and $[ð, ð']$.

Furthermore, we may derive a higher-order decoupled equation governing $\widetilde{\Psi}_{3B}$ from equation (21), using the commutator (5), equations (1), (3), and the Bianchi identity (14). The result is

$$\Big\{(þ - 6\varrho - \overline{\varrho})\Big((þ - 4\varrho - \overline{\varrho})\Big[(þ - 2\varrho - \overline{\varrho})(þ' + 4\mu + \overline{\mu}) - (ð' + 5\pi - \overline{\tau})(ð - \tau + \overline{\pi}) - 3\Psi_2\Big]$$
$$- 3\Big[\Psi_2(2\varrho - \overline{\varrho}) - (ð'\varrho)(ð - \tau + \overline{\pi})\Big]\Big) - 6\varrho\Psi_2(\varrho - \overline{\varrho})\Big\}\widetilde{\Psi}_{3B} = 0. \tag{25}$$

Although it has not yet been found, we expect that a similar, although more complicated, decoupled equation also exists for $\widetilde{\Psi}_{2B}$. We discuss the significance of these higher-order equations in the next section.

### *2.4 Transformations Between The Perturbation Equations*

The Bianchi identities also provide natural transformation identities relating the perturbation equations above. Firstly, we derive the following commutation relation for quantities of $(p, q)$ type $(-4, 0)$

$$\begin{aligned}(þ - 2\varrho - \overline{\varrho})\Big[(þ' + 4\mu + \overline{\mu})(þ - \varrho) - (ð' + 4\pi - \overline{\tau})(ð - \tau) - 3\Psi_2\Big]\\ = \Big[(þ - 2\varrho - \overline{\varrho})(þ' + 4\mu + \overline{\mu}) - (ð' + 5\pi - \overline{\tau})(ð - \tau + \overline{\pi}) - 3\Psi_2\Big](þ - \varrho)\\ -3\Big[\Psi_2(2\varrho - \overline{\varrho}) - (ð'\varrho)(ð - \tau)\Big].\end{aligned} \tag{26}$$

from the complex conjugate of (5) (with $(p, q) = (-3, -1), (a, b) = (\frac{3}{2}, \frac{5}{2})$) and (5) (with $(p, q) = (-4, 0), (a, b) = (3, 0)$). Now, from (14) and (19), and using (26), we see that

$$\begin{aligned}0 = (þ - 2\varrho - \overline{\varrho})\Big[(þ' + 4\mu + \overline{\mu})(þ - \varrho) - (ð' + 4\pi - \overline{\tau})(ð - \tau) - 3\Psi_2\Big]\Psi_{4B}\\ = \Big[(þ - 2\varrho - \overline{\varrho})(þ' + 4\mu + \overline{\mu}) - (ð' + 5\pi - \overline{\tau})(ð - \tau + \overline{\pi}) - 3\Psi_2\Big]\widetilde{\Psi}_{3B}\\ -3\Big[\Psi_2(2\varrho - \overline{\varrho}) - (ð'\varrho)(ð - \tau)\Big]\Psi_{4B}\end{aligned} \tag{27}$$

and thus we derive the perturbation equation for $\widetilde{\Psi}_{3B}$ from (19) by differentiation.

The transformation from $\widetilde{\Psi}_{3B}$ to $\widetilde{\Psi}_{2B}$ follows along similar lines, but is more complicated, so only the result is given here. Applying $(þ - 3\varrho - 2\overline{\varrho})$ to equation (23), using equations (21), (1), (3), (14) and (5) to simplify the right hand side, as well as (5) repeatedly on the left hand side, and after some cancellation, we derive

$$
\begin{aligned}
&\Big[(þ-3\varrho-2\overline{\varrho})(þ'+4\mu+\overline{\mu})-(ð'+5\pi-2\overline{\tau})(ð-2\tau+\overline{\pi})+((þ+\overline{\varrho})\mu-(ð'+\overline{\tau})\tau)\Big](þ-3\varrho)\widetilde{\Psi}_{3B} \\
&\quad = \; 3\varrho\Psi_2\widetilde{\Psi}_{3B}+6\varrho\Psi_2(\varrho-\overline{\varrho})\Psi_{4B}-\widetilde{\Psi}_{3B}(þ-2\overline{\varrho})\Big[(þ+\overline{\varrho})\mu-(ð'+\overline{\tau})\tau\Big] \\
&\qquad +2(ð'\varrho)(\tau+\overline{\pi})\widetilde{\Psi}_{3B}-4(ð'\varrho)(ð-\tau+\overline{\pi})\widetilde{\Psi}_{3B},
\end{aligned}
$$

which becomes the equation (22) for $\widetilde{\Psi}_{2B}$, using (15). Thus we have derived equation (22) from (23) by differentiation.

The transformation from $\Psi_{4B}$ to $\widetilde{\Psi}_{2B}$, which corresponds to the transformation from the Bardeen-Press to the Regge-Wheeler equation in the Schwarzschild case, is achieved by a combination of these transformations. That is

$$\widetilde{\Psi}_{2B}=(þ-3\varrho)(þ-\varrho)\Psi_{4B}. \tag{28}$$

This transformation identity agrees with the results of Sasaki and Nakamura [9, 10] and Chandrasekhar [1], in the sense described below. The essential feature of the transformation is that it consists of two þ operators.

### *2.5 Coordinate Results*

Transformations between the Teukolsky equation and a Regge-Wheeler-like gravitational wave equation in the Kerr background have been investigated thoroughly from a coordinate point of view. The motivation for this investigation is that the Teukolsky equation, while being separable, has long-ranged terms in its effective potential. The analogous question in the Schwarzschild case is the relationship between the Bardeen-Press and Regge-Wheeler equations. Chandrasekhar [1] and Sasaki and Nakamura [9] derived transformations in this case, and subsequently generalised their results to the Kerr background. In [3] we showed how their coordinate results for the Schwarzschild space-time follow in a gauge invariant and geometric manner from the perturbed Bianchi identities in the gauge invariant approach to perturbations, and how they are a part of a much broader picture of perturbations than first imagined.

In the present case, the Kerr analogue of the Regge-Wheeler equation, that is equation (22), is not decoupled or separable. The transformation from the Teukolsky equation to equation (22) is

$$\widetilde{\Psi}_{2B}=(þ-3\varrho)(þ-\varrho)\Psi_{4B}.$$

A natural question to ask is how this result relates to the work of Chandrasekhar [1] and Sasaki and Nakamura [10] on the perturbations of the Kerr space-time. Below we investigate how the essential structure of the transformation identity gives rise to a transformation from the Teukolsky equation to a short-ranged separable Regge-Wheeler-like wave equation, in agreement with the work of Sasaki and Nakamura [10].

We adopt the null tetrad (see Chandrasekhar [1])

$$\begin{aligned} l^\alpha &= \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a) \\ n^\alpha &= \frac{1}{2(r^2+a^2\cos^2\vartheta)}(r^2+a^2, -\Delta, 0, a) \\ m^\alpha &= \frac{1}{(r+ia\cos\vartheta)\sqrt{2}}(ia\sin\vartheta, 0, 1, i\,\mathrm{cosec}\,\vartheta), \end{aligned} \tag{29}$$

with

$$\Delta := r^2 - 2Mr + a^2$$

whereupon

$$\Psi_2 = -\frac{M}{(r-ia\cos\vartheta)^3}$$

or

$$(r - ia\cos\vartheta) = \Big(-\frac{M}{\Psi_2}\Big)^{\frac{1}{3}}.$$

In the absence of the rotation group SO(3) in the background, the complex curvature $\Psi_2$ canonically defines the coordinates $r$ and $\vartheta$ in this way. In the absence of curvature, while both $M$ and $\Psi_2$ vanish individually, their quotient is well behaved. Thus the coordinate results below are also valid in the Schwarzschild and flat space-time cases.

From equations (1) we have that

$$\text{þ}(\Psi_2)^m = 3\varrho m(\Psi_2)^m.$$

Therefore (28) can be written

$$\widetilde{\Psi}_{2B} = \frac{1}{4}(\Psi_2)^{\frac{2}{3}}\text{þþ}\Big(\frac{4(\Psi_2)^{-\frac{4}{3}}\Psi_{4B}}{(\Psi_2)^{-\frac{2}{3}}}\Big).$$

or

$$\widetilde{\Psi}_{2B} = \frac{1}{4\Sigma^2}\text{þþ}\Big(\frac{4\Sigma^4\Psi_{4B}}{\Sigma^2}\Big) \tag{30}$$

where

$$\Sigma := r - ia\cos\vartheta.$$

Define the new field quantity

$$R_{-2} := 4\Sigma^5\widetilde{\Psi}_{2B}. \tag{31}$$

In the Schwarzschild limit, the radial part of this quantity corresponds to the Regge-Wheeler field, satisfying the Regge-Wheeler wave equation in coordinates after expanding in spin-weighted spherical harmonics.

The transformation identity (30) becomes

$$R_{-2} = \Sigma^3\mathcal{D}_0\mathcal{D}_0\Big(\frac{T_{-2}}{\Sigma^2}\Big) \tag{32}$$

where

$$T_{-2} = 4\Sigma^4 \Psi_{4B} \tag{33}$$

is the usual Teukolsky quantity, and

$$\mathcal{D}_0 := \frac{r^2 + a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi.$$

While $T_{-2}$ is separable in the sense described in Teukolsky [14] and Chandrasekhar [1], $R_{-2}$ is not separable.

Introducing the slightly modified quantity

$$R^*_{-2} := \Sigma^3_* \mathcal{D}_0 \mathcal{D}_0 \left(\frac{T_{-2}}{\Sigma^2_*}\right) \tag{34}$$

where $\Sigma_* = \sqrt{r^2 + a^2}$, we see that $R^*_{-2}$ is separable, and it coincides with (32) when $a \to 0$. In fact, in the Schwarzschild limit, (34) compares very favourably with our Schwarzschild result [3] equation (3.85), and, as discussed at length in [3], agrees with the transformation provided by Sasaki and Nakamura [10] and Chandrasekhar [1]. Now, the relation (34) is precisely the one which Sasaki and Nakamura found from another point of view (see Sasaki and Nakamura [10] equation (2.13), with $f = g = h = 1$). As shown by Sasaki and Nakamura, the quantity $R^*_{-2}$ satisfies a (homogeneous) Regge-Wheeler-like differential equation (with short-ranged effective potential) in the Kerr case, after separating the variables in the usual way. Now, if it were possible to reconstruct $\Psi_{4B}$ from $R^*_{-2}$, we would have a complete determination of our gauge invariant field quantities in terms of $R^*_{-2}$. In this case, solving the Regge-Wheeler-like equation determines all of the natural gauge invariant field quantities.

The analysis presented here can be applied in the case of the electromagnetic perturbations of the Kerr space-time as well, and the results are presented elsewhere [11]. It is in this sense that the Fackerell-Ipser [18] equation for the electromagnetic perturbations can be made separable. The resulting equation reduces to the Regge-Wheeler equation for electromagnetic perturbations in the Schwarzschild case.

## 3. Reissner-Nordström Space-time

Our attention now turns to the Reissner-Nordström space-time. Using the null tetrad:

$$\begin{aligned} l^\alpha &= \frac{1}{\Delta}(r^2, \Delta, 0, 0) \\ n^\alpha &= \frac{1}{2r^2}(r^2, -\Delta, 0, 0) \\ m^\alpha &= \frac{1}{r\sqrt{2}}(0, 0, 1, i \operatorname{cosec} \vartheta) \end{aligned} \tag{35}$$

where $\Delta = r^2 - 2Mr + Q^2$, we have

$$\kappa = \sigma = \lambda = \nu = \varepsilon = \pi = \tau = \Phi_0 = \Phi_2 = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$$

$$\text{þ}\varrho = \varrho^2,\ \text{þ}'\varrho = -\varrho\mu - \Psi_2,\ \text{þ}\mu = \varrho\mu + \Psi_2,\ \text{þ}'\mu = -\mu^2$$
$$\text{þ}\Psi_2 = (3\Psi_2 + 2\Phi_{11})\varrho,\ \text{þ}'\Psi_2 = -(3\Psi_2 + 2\Phi_{11})\mu,\ \text{þ}\Phi_1 = 2\varrho\Phi_1,\ \text{þ}'\Phi_1 = -2\mu\Phi_1$$
$$\eth\Psi_2 = \eth\Phi_1 = \eth\mu = \eth\varrho = \eth'\Psi_2 = \eth'\Phi_1 = \eth'\mu = \eth'\varrho = 0 \tag{36}$$

and

$$\varrho = \overline{\varrho} = -\frac{1}{r},\ \mu = \overline{\mu} = -\frac{\Delta}{2r^3},\ \alpha = -\beta = -\frac{\cot\vartheta}{2r\sqrt{2}},$$
$$\gamma = \frac{(Mr - Q^2)}{2r^3},\ \Psi_2 = -\frac{Mr - Q^2}{r^4},\ \Phi_1 = \frac{Q}{2r^2},\ \Phi_{11} = 2{\Phi_1}^2. \tag{37}$$

Thus,

$$r = \left(\frac{Q}{2\Phi_1}\right)^{\frac{1}{2}}$$
$$\text{þ}r = -\varrho r,\ \text{þ}'r = \mu r,\ \eth r = 0. \tag{38}$$

In this article, we use the convention $\Phi_{mn} = 2\Phi_m\overline{\Phi}_n$.

In Reissner-Nordström background, the perturbed Maxwell equations take the form:[5]

$$\eth'\Phi_{1B} + (\eth' + 2\pi)_B\Phi_1 = (\text{þ} - \varrho)\Phi_{2B} \tag{39}$$
$$(\text{þ}' + 2\mu)\Phi_{1B} + (\text{þ}' + 2\mu)_B\Phi_1 = \eth\Phi_{2B} \tag{40}$$
$$\eth\Phi_{1B} + (\eth - 2\tau)_B\Phi_1 = (\text{þ}' + \mu)\Phi_{0B} \tag{41}$$
$$(\text{þ} - 2\varrho)\Phi_{1B} + (\text{þ} - 2\varrho)_B\Phi_1 = \eth'\Phi_{0B}. \tag{42}$$

We will use the following two perturbed Bianchi identities:

$$(\text{þ}' + 3\mu)\Psi_{2B} + (\text{þ}' + 3\mu)_B\Psi_2 = \eth\Psi_{3B} + 2\Phi_1\eth\Phi_{2B} - 4\mu\Phi_1(\overline{\Phi}_{1B} + \Phi_{1B}) - 2\mu_B\Phi_{11} \tag{43}$$
$$(\text{þ} - 2\varrho)\Psi_{3B} = \eth'\Psi_{2B} + (\eth' + 3\pi)_B\Psi_2 + 2\Phi_1\text{þ}\Phi_{2B} + 4\mu\Phi_1\overline{\Phi}_{0B} - 2\pi_B\Phi_{11} \tag{44}$$

and the Ricci identity:

$$\text{þ}\mu_B + \text{þ}_B\mu - \eth\pi_B = \overline{\varrho}_B\mu + \overline{\varrho}\mu_B + \Psi_{2B}. \tag{45}$$

Unlike in the vacuum cases, both electromagnetic and gravitational quantities appear in the Maxwell equations and in the Bianchi identities. Using the derivatives of $\Phi_1$ and $\Psi_2$ from (36), and since $\varrho = \overline{\varrho}$, $\mu = \overline{\mu}$ in this case, the perturbed commutators $\left([\eth, \eth']\Phi_1\right)_B$ and $\left([\eth, \eth']\Psi_2\right)_B$ can be rewritten

$$\left[\eth\eth'_B - \eth'\eth_B\right]\Phi_1 = 2\Phi_1\left[\varrho(\mu_B - \overline{\mu}_B) - \mu(\varrho_B - \overline{\varrho}_B)\right] \tag{46}$$
$$\left[\eth\eth'_B - \eth'\eth_B\right]\Psi_2 = (3\Psi_2 + 2\Phi_{11})\left[\varrho(\mu_B - \overline{\mu}_B) - \mu(\varrho_B - \overline{\varrho}_B)\right]. \tag{47}$$

[5] Chandrasekhar [1], Chapter 1, equations (330)–(333).

*3.1 Gauge Invariant Perturbation Equations*

From the Maxwell equation (40), and its complex conjugate, we have

$$(þ' + 2\mu)(\Phi_{1B} - \overline{\Phi}_{1B}) + 2\Phi_1(\mu_B - \overline{\mu}_B) = ð\Phi_{2B} - ð'\overline{\Phi}_{2B} \tag{48}$$

since $\Phi_1 = \overline{\Phi}_1$ and $þ'_B = \overline{þ'_B}$. Operating on this with $(þ - 2\varrho)$ and using the Newman-Penrose commutator $[þ, ð]$ in conjunction with equations (36), we see that

$$(þ - 2\varrho)(þ' + 2\mu)(\Phi_{1B} - \overline{\Phi}_{1B}) + 2\Phi_1 þ(\mu_B - \overline{\mu}_B) = ð(þ - \varrho)\Phi_{2B} - ð'(þ - \varrho)\overline{\Phi}_{2B}. \tag{49}$$

From equation (39) and its complex conjugate, and using $[ð, ð']$, the right hand side of equation (49) may be expanded to

$$ðð'(\Phi_{1B} - \overline{\Phi}_{1B}) + 2\Phi_1(ð\pi_B - ð'\overline{\pi}_B) + (ðð'_B - ð'ð_B)\Phi_1.$$

Thus, equation (49) may be written

$$\Big[(þ-2\varrho)(þ'+2\mu)-ðð'\Big](\Phi_{1B}-\overline{\Phi}_{1B}) = -2\Phi_1\Big[þ(\mu_B-\overline{\mu}_B)-(ð\pi_B-ð'\overline{\pi}_B)\Big]+(ðð'_B-ð'ð_B)\Phi_1. \tag{50}$$

Now, from the Ricci identity (45) and its complex conjugate,

$$þ(\mu_B - \overline{\mu}_B) - ð\pi_B + ð'\overline{\pi}_B = -\mu(\varrho_B - \overline{\varrho}_B) + \varrho(\mu_B - \overline{\mu}_B) + (\Psi_{2B} - \overline{\Psi}_{2B}). \tag{51}$$

Substituting (46) and (51) into the right hand side of (50), we derive the following perturbation equation for $\Phi_{1B} - \overline{\Phi}_{1B}$

$$\Big[(þ - 2\varrho)(þ' + 2\mu) - ðð'\Big](\Phi_{1B} - \overline{\Phi}_{1B}) = -2\Phi_1(\Psi_{2B} - \overline{\Psi}_{2B}). \tag{52}$$

When $\Phi_1 = 0$, we recover precisely the Regge-Wheeler equation for electromagnetic fields in the Schwarzschild background.

In a similar calculation, starting from the perturbed Bianchi identities and using the perturbed Maxwell equations, we may derive the following wave-type equation for the gravitational field.

$$\begin{aligned}\Big[(þ - 3\varrho)(þ' + 3\mu) - ðð' + 3\Psi_2 + 2\Phi_{11})\Big](\Psi_{2B} - \overline{\Psi}_{2B})&\\ = 4\Phi_1\Big[ðð' + \varrho þ' - \mu þ + 4\varrho\mu\Big](\Phi_{1B} - \overline{\Phi}_{1B}).&\end{aligned} \tag{53}$$

Equations (52) and (53) may be derived in the context of extending the gauge invariant formalism, presented in the previous section, to the Reissner-Nordström case (Fernandes [19], Fernandes and Lun [13]), as well as in the naive fashion presented above from the Newman-Penrose equations.

Importantly, the perturbed field quantities $Im(\Psi_{2B})$ and $Im(\Phi_{1B})$ are invariant under infinitesimal coordinate transformations and infinitesimal Lorentz transformations. The proof of this statement requires a brief calculation, but follows directly from Lun [4]. Alternatively, since $\Psi_2 = \overline{\Psi}_2$ and $\Phi_1 = \overline{\Phi}_1$ then the Lie derivatives of $Im(\Psi_2)$ and $Im(\Phi_1)$

vanish. So $Im(\Psi_{2B})$ and $Im(\Phi_{1B})$ are coordinate-gauge invariant, and they are also unaffected by infinitesimal tetrad transformations.

In addition, when $\Phi_1 = 0$ (or $Q = 0$) we recover precisely the Regge-Wheeler equations for the electromagnetic and gravitational perturbations of the Schwarzschild space-time. Thus, equations (52) and (53) are the generalisation of Price's [20] result for the perturbations of the Schwarzschild space-time to the case of the Reissner-Nordström black hole. These equations may be decoupled, providing a pair of Regge-Wheeler-type equations for the gauge invariant perturbations of Reissner-Nordström space-time. In coordinates, for an $l$-pole, we have

$$\Big[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{\Delta}{r^5}\Big(l(l+1)r - 6M + \frac{4Q^2}{r}\Big)\Big] \begin{pmatrix} Z_+ \\ Z_- \end{pmatrix} = \frac{2\Delta}{r^5} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \begin{pmatrix} Z_+ \\ Z_- \end{pmatrix} \tag{54}$$

where

$$\begin{pmatrix} Z_+ \\ Z_- \end{pmatrix} := \begin{pmatrix} 1 & \frac{1}{2}\Big(\frac{2}{Q}\Big)^{\frac{3}{2}}\lambda_+ \\ 1 & \frac{1}{2}\Big(\frac{2}{Q}\Big)^{\frac{3}{2}}\lambda_- \end{pmatrix} \begin{pmatrix} r^3(\Psi_{2B} - \overline{\Psi}_{2B}) - 2rQ(\Phi_{1B} - \overline{\Phi}_{1B}) \\ r^2\Big(\frac{Q}{2}\Big)^{\frac{1}{2}}(\Phi_{1B} - \overline{\Phi}_{1B}) \end{pmatrix}$$

$$\lambda_\pm = \frac{3M}{2}\Bigg(1 \pm \sqrt{1 + \frac{4Q^2(l-1)(l+2)}{9M^2}}\Bigg).$$

In fact this is an alternative derivation, from the point of view of the modified Newman-Penrose formalism, of Moncrief's [15] equations, which he derived for the "odd-parity" perturbations of the Reissner-Nordström space-time using the Hamiltonian formulation. A detailed derivation of this result is given elsewhere [12].

Finally, a decoupled higher-order equation governing the electromagnetic perturbations can be obtained directly from equation (52). Applying the operator

$$\Big[(\text{þ} - 5\varrho)(\text{þ}' + 5\mu) - \eth\eth' + 3\Psi_2 + 2\Phi_{11}\Big]$$

to equation (52) and rewriting the right hand side according to (53), we derive the following fourth-order (gauge invariant) equation for $(\Phi_{1B} - \overline{\Phi}_{1B})$:

$$\begin{aligned}\Big[(\text{þ} - 5\varrho)(\text{þ}' + 5\mu) - \eth\eth' + 3\Psi_2 + 2\Phi_{11}\Big]\Big[(\text{þ} - 2\varrho)(\text{þ}' + 2\mu) - \eth\eth'\Big](\Phi_1 - \overline{\Phi}_1)_B \\ = \quad - 4\Phi_{11}(\eth\eth' + \varrho\text{þ}' - \mu\text{þ} + 4\varrho\mu)(\Phi_1 - \overline{\Phi}_1)_B\end{aligned} \tag{55}$$

or, expanding in coordinates for an $l$-pole

$$\begin{aligned}\Big[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{\Delta}{r^5}\Big(l(l+1)r - 6M + \frac{8Q^2}{r}\Big)\Big]\frac{r^5}{\Delta}\Big[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{\Delta}{r^4}l(l+1)\Big)\Big] r^2(\Phi_{1B} - \overline{\Phi}_{1B}) \\ = -\frac{4\Delta Q^2}{r^5}\Big[2r\frac{\partial}{\partial r_*} - l(l+1)\Big] r^2(\Phi_{1B} - \overline{\Phi}_{1B}).\end{aligned} \tag{56}$$

This equation is fourth-order in derivatives of time, and hence may include solutions which are unphysical. However, since we are not required to take the square root of angular derivatives, as occurs in the decoupling of equations (52) and (53), there are some advantages in investigating higher-order equations such as equations (56). In particular this will be significant when the background geometry does not give rise to separable wave equations, as is the case in the Kerr space-time (c.f. equation (25)). A similar fourth-order equation may exist for the gravitational perturbations, although this has not yet been found.

## 4. Discussion

The perturbations of the Kerr and Reissner-Nordström space-times can be investigated using a generalization of the gauge invariant approach developed for the Schwarzschild case. In the present article we have presented some of the results of this analysis.

In the Kerr space-time the extension of the techniques used in the Schwarzschild case is purely technical. The perturbed Bianchi identities are rewritten in a form involving only certain natural gauge invariant perturbed field quantities. We then derive a system of gauge invariant perturbation equations from the perturbed Bianchi identities, as well transformations which link each perturbation equation to each other. These results can be seen to agree with the transformations derived by Sasaki and Nakamura [9, 10] and Chandrasekhar [1], between the Teukolsky equation and a Regge-Wheeler-like equation. The essential feature of this transformation is that it consists of a pair of radial differential operators, after specifying the time and angular dependence.

As stated above, the success of the gauge invariant technique is due to the geometry of the background space-time. Importantly, much of the structure of the perturbed Bianchi identities in the Schwarzschild background is also present in the Kerr case. The only added complication is due to the presence of angular momentum.

With angular momentum present, the background is not spherically symmetric, and we are not able to expand the perturbed fields in harmonics in general. Consequently our gauge invariant field quantities each have spin-weight $-2$. Furthermore, the (second-order wave-like) gravitational perturbation equations are not decoupled, except for the well known Teukolsky equation. In addition, whereas $\Psi_{4B}$ can be made separable in the usual sense by multiplication by $(r - ia\cos\vartheta)^4$ (see Teukolsky [14], Chandrasekhar [1]), the same is not true for $\widetilde{\Psi}_{3B}$ or $\widetilde{\Psi}_{2B}$. This can be seen clearly, for example, when $\widetilde{\Psi}_{2B}$ is written in terms of $\Psi_{4B}$ in equation (32), and the issue has been addressed by Sasaki and Nakamura [9, 10] and Chandrasekhar [1], as well as in the text above. Nevertheless, each equation reduces to its correct Schwarzschild form when $\pi = 0$, $\Psi_2 = \overline{\Psi}_2$ (which corresponds to $a = 0$).

Given the success of the gauge invariant technique in treating the perturbations of the Schwarzschild and Kerr space-times, our attention turns to the electrovac cases. The presence of charge in the background complicates the perturbation problem greatly. This can be seen in the complex nature of the Maxwell equations (39)–(42) and the Bianchi identities (43) and (44) in Reissner-Nordström background, where gravitational and electromagnetic field quantities mix (seemingly) freely. The background charge leads to the nontrivial coupling between the gravitational and electromagnetic fields. This coupling arises in the first instance in the perturbed Maxwell equations, through directional derivatives of the Coulomb field along the perturbed tetrad directions, and in the perturbed Bianchi identities, through the Newman-Penrose components of the Ricci tensor. We notice also that the perturbed Bianchi identities involve the complex conjugate of quantities of opposite spin-weight, in contrast to the Bianchi identities in the vacuum backgrounds. So the structure of the Newman-Penrose equations changes.

However, the techniques presented for the vacuum cases do generalise to the electrovac space-times. Above we have shown how gauge invariant decoupled perturbation equations arise from the perturbed Newman-Penrose equations. These results can be achieved in the context of the gauge invariant formalism as well as in the fashion described in this article. The equations are a generalisation of Price's [20] result for the perturbations of the Schwarzschild space-time, and also provide an alternative derivation, from the point of view of the modified Newman-Penrose formalism, of Moncrief's [15] equation for the "odd-parity" perturbations of the Reissner-Nordström black hole. These results are a precursor to extending the gauge invariant techniques to the Reissner-Nordström space-time.

In the full gauge invariant treatment of the perturbations of the Reissner-Nordström space-time (see [13], [19]), we have been able to identify the natural gauge invariant electromagnetic and gravitational field quantities. When they are rewritten in a form involving only these quantities, the integrability conditions for the perturbed Maxwell equations and Bianchi identities provide a rather complicated system of coupled perturbation equations. A decoupling of these equations will follow from a detailed consideration of the role of charge in the background space-time. It is expected that the equations will simplify greatly when appropriate combinations of the field quantities are taken, and these may involve mixing field quantities of opposite spin-weight, as occurs in the Bianchi identities in this case. The results in the Reissner-Nordström case throw some light on the complicated nature of the coupling of the electromagnetic and gravitational fields, without the added complexity of angular momentum in the background space-time.

Our ultimate objective is to extend our gauge invariant approach to the perturbations of the Kerr-Newman space-time. Although the perturbations of the Kerr-Newman space-time have defied many attempts at clarification previously, our preliminary results indicate that the gauge invariant technique may be applied in that case as well. The main problem to be addressed is the combined effect of angular momentum and charge in the background. It is hoped that the present work will lead to a deeper understanding of the role played by angular momentum in the perturbations of vacuum space-times, and charge in the spherically symmetric case, and will thus enable us to extend our approach to the perturbations of the Kerr-Newman space-time.

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