Square of general relativity

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Abstract

We consider dilaton–axion gravity interacting with p $U(1)$ vectors ($p = 6$ corresponding to $N = 4$ supergravity) in a four–dimensional spacetime admitting a non–null Killing vector field. It is argued that this theory exhibits features of a "square" of vacuum General Relativity. In the three–dimensional formulation it is equivalent to a gravity coupled σ–model with the $(4 + 2p)$–dimensional target space $SO(2, 2+p)/(SO(2) \times SO(2+p))$. Kähler coordinates are introduced on the target manifold generalising the Ernst potentials of General Relativity. The corresponding Kähler potential is found to be equal to the logarithm of the product of the four–dimensional metric component g_{00} in the Einstein frame and the dilaton factor, independently of the presence of vector fields. The Kähler potential is invariant under exchange of the Ernst potential and the complex axidilaton field, while it undergoes holomorphic/antiholomorphic transformations under general target space isometries. The "square" property is also manifest in the two–dimensional reduction of the theory as a matrix generalisation of the Kramer–Neugebauer map.

Supergravity is often called a "square root" of General Relativity. Indeed, a supersymmetric extension of the Poincaré algebra is reminiscent of the Dirac's procedure of obtaining a spin 1/2 wave equation from the scalar wave equation. A bosonic sector of *extended* supergravities, apart from the graviton, contains scalar and vector fields. One of the most interesting bosonic structures is suggested by the $N = 4$ supergravity, which attracted much attention recently in connection with "stringy" black holes [1, 2, 3, 4]. Here we want to discuss the relationship between this theory (often called also dilaton–axion gravity) and vacuum General Relativity and to show that, in a certain sense, it can be viewed as a "square" of the latter.

As far as *stationary* solutions such as black holes are concerned, (or more generally, space–times possessing a non–null Killing vector field), Einstein's theory may be reformulated as a three–dimensional gravity coupled non–linear σ–model [5]. This theory admits a concise representation in terms of the Ernst potential ϵ [6, 7], which may be regarded as a complex coordinate on a one–dimensional Kähler manifold $SU(1,1)/U(1)$. A similar representation holds for the dilaton–axion σ–model. The Kähler potential for the target space of the three–dimensional $N = 4$ supergravity turns out to be equal to the logarithm of the product of potentials of the vacuum gravity and the dilaton–axion system independently of the presence of vector fields. For vacuum gravity this potential is simply given by the logarithm of the four–dimensional g_{00}. A dilaton exponential plays a similar role in the dilaton–axion σ–model possessing the same $SU(1,1)/U(1)$ structure. A square of this coset generates the target space of the pure dilaton–axion gravity, which is enlarged to the manifold $SO(2,2+p)/(SO(2)\times SO(2,p))$ when vector fields are included. However, the Kähler potential still preserves its value given by the pure dilaton–axion gravity. This gives rise to various similarities between classical solutions of the Einstein–Maxwell–dilaton–axion theory and vacuum Einstein equations. One can say that $N=4$ supergravity exhibits at the same time features of the square root and the square of General Relativity. Apart from this somewhat philosophical implication, the Kähler representation of the stationary $N=4$ supergravity turns out to be very useful in identifying hidden symmetries and generating classical solutions.

Let us recall first the Ernst formulation of the stationary vacuum Einstein equations [6]. Assuming the spacetime to admit a timelike (in an essential region) Killing vector field, one parameterises the four–dimensional metric through the standard Kaluza–Klein ansatz

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(dt-\omega_i dx^i)^2 - \frac{1}{f}h_{ij}dx^i dx^j, \tag{1}$$

where the three–space metric, h_{ij}, $(i,j=1,2,3)$, the rotation one–form, ω_i and the three–dimensional conformal factor, f, depend on the three–space coordinates x^i only. It can then be shown that the vanishing of the mixed components of the Ricci tensor $R^i{}_0$ implies the existence of the NUT–potential χ replacing a rotation one–form $\omega=\omega_i dx^i$ via dualisation [8]

$$d\chi = -f^2 * d\omega, \tag{2}$$

where an asterisk stands for the three–dimensional Hodge dual. Together f and χ parameterise a two–dimensional target manifold of the σ–model resulting from dimensional reduction. A Kähler representation is achieved by introducing the following linear combination, the (vacuum) Ernst potential,

$$\epsilon = if - \chi. \tag{3}$$

(The more frequently used potential differs from this one by a factor i.) The corresponding equations of motion define a harmonic map from the three–dimensional space x^i to the target manifold endowed with the metric

$$dl^2 = 2G_{\epsilon\bar\epsilon}d\epsilon d\bar\epsilon = -2\frac{d\epsilon d\bar\epsilon}{(\bar\epsilon-\epsilon)^2}. \tag{4}$$

The Kähler metric $G_{\epsilon\bar\epsilon}$ is generated by the Kähler potential K

$$G_{\epsilon\bar\epsilon} = \partial_\epsilon\partial_{\bar\epsilon}K(\epsilon,\,\bar\epsilon), \tag{5}$$

for which one obtains the following simple expression

$$K = -\ln V, \quad V = \mathrm{Im}\epsilon = f. \tag{6}$$

Thus the Kähler potential for any stationary solution of the vacuum Einstein equation is directly related to the g_{00} component of the four–dimensional metric.

The Ernst potential acts as a source in the three–dimensional Einstein equations for the metric h_{ij}

$$\mathcal{R}_{ij} = \frac{1}{2}\epsilon_{(i}\bar{\epsilon}_{j)}(\mathrm{Im}\epsilon)^{-2}. \tag{7}$$

Once a solution is found, to restore a four–dimensional metric one has merely to solve a linear equation (2) for the rotation one–form ω. To solve three–dimensional gravity coupled σ–model equations additional assumptions are needed in general, such as existence of the second space–time Killing field commuting with the first one. In this case, a further reduction to two dimensions leads to completely integrable (modified) chiral equations with associated infinite symmetries [9]. Another useful technique consists of restricting the functional dependence of all target variables on space coordinates through a unique scalar function (or several such functions) [10]. However, information already contained in the structure of the target manifold may be helpful in generating new solutions from already known ones. This amounts to using the target space isometries to relate between themselves physically inequivalent field configurations. It is then important to find a concise representation of symmetry transformations in terms of physical quantities. In General Relativity it was precisely the Ernst formulation which allowed for considerable simplifications. It is natural therefore to look for similar representation in more general supergravity inspired bosonic theories.

The isometry group of the target manifold is a global symmetry of the system, which maps one classical solution to another. For the vacuum Einstein system (4) it can be written in the $SL(2,R)$ form

$$\epsilon \to \frac{a\epsilon + b}{c\epsilon + d}, \quad ad - bc = 1, \tag{8}$$

with real a, b, c, d. This three–parametric group can be conveniently cast into three one–parameter subgroups. The first subgroup is the shift of the complex Ernst potential by a real constant

$$i) \qquad \epsilon \to \epsilon + b, \quad (a = d = 1,\ c = 0). \tag{9}$$

This changes the NUT potential by a constant, which in view of (2) does not modify the metric. This transformation is thus a pure gauge.

The second subgroup is the rescaling of the Ernst potential

$$ii) \qquad \epsilon \to a^2\epsilon, \quad (b = c = 0,\ d = 1/a). \tag{10}$$

This preserves the r.h.s. of the three–dimensional Einstein equations (7), but modifies the four–dimensional metric (1), thus producing physically inequivalent field configurations (in particular, transforming asymptotically flat solutions into asymptotically non–flat ones).

To write down the third subgroup one has first to make a discrete transformation: an inversion of the Ernst potential $\epsilon \to \epsilon^{-1}$ (which corresponds to $a = d = 0,\ b = -c = 1$ in (8) plus change of a sign). After that one also makes a shift by a real constant:

$$iii) \qquad \epsilon^{-1} \to \epsilon^{-1} + c, \quad (d = a = 1,\ b = 0). \tag{11}$$

This is the Ehlers transformation [11], an essential part of the whole group. It is non–linear being expressed in terms of ϵ, and physically corresponds to mixing of a mass and a NUT charge (a gravitational analogue of electric–magnetic duality).

The remarkable property of this σ–model is that the target manifold is a *symmetric* Riemannian space. This property opens up prospects of obtaining an infinite–dimensional symmetry (Geroch group) [9] if one further assumes the existence of the second space–time isometry commuting with the first one (stationary axisymmetric fields, plane waves, non–homogeneous cosmologies etc.). The corresponding symmetry group will be then an affine extension of $SL(2,R)$. Some of its lower–level elements were obtained in various explicit forms as Bäcklund transformations [12], and were used to generate new exact solutions of Einstein equations. It is worth noting that the most sophisticated known exact solutions belong to this type.

Obviously, the hidden symmetry of the stationary vacuum Einstein equations is identical to the S–duality [13] of the dilaton–axion system (also $SL(2,R) \sim SU(1,1)$), which is a part of the bosonic sector of $N=4$, $D=4$ supergravity. Let us consider first the *pure* dilaton–axion system coupled to gravity. (Previous discussion of this model can be found in [14, 15].) Denoting the four–dimensional Peccei–Quinn axion as κ and introducing a complex axidilaton field,

$$z = \kappa + i\mathrm{e}^{-2\phi}, \tag{12}$$

one can write the action as follows

$$S = \int \left\{ -R + 2\left|\partial z(z-\bar{z})^{-1}\right|^2 \right\} \sqrt{-g} d^4x. \tag{13}$$

Clearly the axion–dilaton term has exactly the same symmetries (8) with ϵ replaced by z. The role of the gauge transformation now is played by the shift of the axion by a constant, while the inversion is related to the strong–weak coupling duality transformation.

These symmetries survive upon reduction to three dimensions. If the condition of stationarity (1) is imposed, the target manifold of the resulting three–dimensional σ–model will be the product of two copies of $SU(1,1)/U(1)$, one in terms of the Ernst potential, and another in terms of z:

$$dl^2 = 2G_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta = -2\left\{ \frac{d\epsilon d\bar{\epsilon}}{(\bar{\epsilon}-\epsilon)^2} + \frac{dz d\bar{z}}{(\bar{z}-z)^2} \right\}. \tag{14}$$

Now the Kähler metric $G_{\alpha\bar{\beta}}$, $(\alpha, \beta = 0, 1)$ is generated by the potential

$$G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z^\alpha, \bar{z}^\beta), \quad z^\alpha = (\epsilon,\, z), \tag{15}$$

$$K = -\ln V, \quad V = \mathrm{Im}\epsilon\; \mathrm{Im}z = f\mathrm{e}^{-2\phi}. \tag{16}$$

The V–potential is given by the product of four–dimensional g_{00} and the dilaton factor, which plays similar role in the target space geometry. We observe therefore that the stationary dilaton–axion gravity has the remarkable property of a "square" of vacuum gravity. The reason for this is simply that the three–dimensional reductions of both the vacuum Einstein gravity and the dilaton–axion system have identical σ–model representations. The coupled system also possesses the "Ernst–axidilaton" duality symmetry under an exchange

$$\epsilon \leftrightarrow z. \tag{17}$$

The situation becomes slightly more complicated when vector fields are included. In three dimensions vector fields can be traded for scalars, and one can expect to get a larger sigma–model with a higher–dimensional target manifold [16]. This is indeed the case, e.g., for the Einstein–Maxwell theory [5, 17] (bosonic sector of $N = 2$ supergravity), where one obtains a Kähler target manifold $SU(2,1)/(SU(1,1)\times U(1))$, as well as for other supergravities and dimensionally reduced Kaluza–Klein theories [16]. We will discuss now the $N = 4$ supergravity containing a dilaton, an axion, and six abelian vector fields (for the sake of generality we take p vector fields). It turns out that the target manifold is also Kähler, and the corresponding complex coordinates are some generalisations of the Ernst potentials of the Einstein–Maxwell theory. For the model with only one vector field such coordinates were recently found [18, 19] to provide a convenient parameterisation for the Ehlers–Harrison transformations of the theory, which had been discovered earlier [3] in terms of real variables. When several vector fields are present, the target space is extended rather straightforwardly.

Consider a four–dimensional action

$$S = \int \left\{ -R + 2\left|\partial z (z-\bar{z})^{-1}\right|^2 + \left(iz\mathcal{F}^n_{\mu\nu}\mathcal{F}^{n\mu\nu} + c.c \right) \right\} \sqrt{-g}d^4x, \tag{18}$$

where $\mathcal{F}^n = (F^n + i\tilde{F}^n)/2$, $\tilde{F}^{n\mu\nu} = \frac{1}{2}E^{\mu\nu\lambda\tau}F^n_{\lambda\tau}$, $n = 1, ..., p$, and the sum over repeated n is understood. For $p = 6$ this is the bosonic sector of $N = 4, D = 4$ supergravity. This action is invariant under $SO(p)$ rotations of vector fields, which is an analogue of T–duality of dimensionally reduced theories [20]. The equations of motion and Bianchi identities (but not the action) are also invariant under S–duality transformations

$$z \to \frac{az+b}{cz+d}, \quad ad - bc = 1,$$

$$F^n \to (c\kappa + d)F^n + c\mathrm{e}^{-2\phi}\tilde{F}^n. \tag{19}$$

Imposing the stationarity condition (1) one can express vector fields through the electric v^n and magnetic u^n scalar potentials as follows

$$F^n_{i0} = \frac{1}{\sqrt{2}}\partial_i v^n, \tag{20}$$

$$2\mathrm{Im}\left(z\mathcal{F}^{nij}\right) = \frac{f}{\sqrt{2h}}\epsilon^{ijk}\partial_k u^n. \tag{21}$$

In three dimensions the rotation one form ω_i plays the role of a graviphoton, and one can show using the standard argument that the "T–duality" group is enlarged to $SO(1, p+1)$. Also, S–duality becomes the symmetry of the three–dimensional *action*. Moreover, both these groups turn out to be unified in a larger "U–duality" group $SO(2, p+2)$ [3, 19, 21]. This can be easily checked by computing the Kähler metric of the resulting target manifold. To find the σ–model representation one has to introduce a NUT potential χ via

$$d\chi = u^n dv^n - v^n du^n - f^2 * d\omega, \tag{22}$$

and to derive the set of equations for χ, u^n in addition to the equations for f, κ, ϕ, v^n. The full set of equations will be that of the three–dimensional gravity coupled σ–model possessing the $4+2p$ dimensional target space $SO(2, 2+p)/\left(SO(2) \times SO(p, 2)\right)$. One can parameterise the target manifold by complex coordinates z^α, $\alpha = 0, 1, ..., p+1$ which have the following meaning. The components $\alpha = n = 1, ..., p$ are complex potentials for vector fields

$$z^n = u^n - zv^n \equiv \Phi^n, \quad n = 1, ..., p, \tag{23}$$

while the $\alpha = p+1$ component is the complex axidilaton field itself, $z^{p+1} = z$, and

$$z^0 = \epsilon + v^n \Phi^n \equiv E, \tag{24}$$

is the $N = 4$ analogue of the Ernst potential. Somewhat surprisingly, the Kähler potential, generating the target space metric via (15), remains untouched by the electric and magnetic potentials and preserves its value (16) given by pure dilaton–axion gravity:

$$K = -\ln V, \quad V = \mathrm{Im}E\ \mathrm{Im}z + (\mathrm{Im}\Phi^n)^2 = f\mathrm{e}^{-2\phi}. \tag{25}$$

Hence, in a sense, the "square" property of the pure dilaton–axion gravity is not destroyed by vectors. At the same time, being expressed through complex coordinates, the Kähler potential has a non–trivial dependence on all of them, so that the metric of the target space is non–degenerate.

Since the Kähler metric (5) is given by mixed derivatives of holomorphic and anti-holomorphic coordinates, a multiplication of V by an arbitrary holomorphic function and its complex conjugate (to preserve reality of V) does not change the metric. Thus a transformation

$$V(z,\ \bar{z}) \to f(z)\bar{f}(\bar{z})V(z,\ \bar{z}) \tag{26}$$

is a target space isometry. The Ernst–axidilaton duality (17) (with Φ^n unchanged) belongs trivially to this class. Another useful discrete symmetry corresponds to

$$f(z) = \left(Ez + \mathbf{\Phi}^2\right)^{-1}, \quad \mathbf{\Phi}^2 \equiv \Phi^{n\,2}, \tag{27}$$

and consists of the following:

$$E \to \frac{z}{Ez + \mathbf{\Phi}^2}, \quad z \to \frac{E}{Ez + \mathbf{\Phi}^2}, \quad \mathbf{\Phi} \to \frac{\mathbf{\Phi}}{Ez + \mathbf{\Phi}^2}. \tag{28}$$

Three–dimensional U–duality transformations $SO(2, 2+p)$ of $N = 4$ supergravity can now be listed in the following way. The most obvious symmetries include $p(p-1)/2$ $SO(p)$

rotations acting only on vector fields, $\mathbf{\Phi} \to \mathbf{\Omega}\mathbf{\Phi}$, where $\mathbf{\Omega}^T\mathbf{\Omega} = I_p$, as well as $2p+1$ gauge transformations

$$\text{gravitational}: \quad E \to E + g, \qquad \mathbf{\Phi}, z \text{ unchanged}, \tag{29}$$

$$\text{magnetic}: \quad \mathbf{\Phi} \to \mathbf{\Phi} + \mathbf{m}, \qquad E, z \text{ unchanged}, \tag{30}$$

$$\text{electric}: \quad \mathbf{\Phi} \to \mathbf{\Phi} + \mathbf{e}z, \quad E \to E - 2\mathbf{e}\mathbf{\Phi} - \mathbf{e}^2 z, \qquad z \text{ unchanged}, \tag{31}$$

and scale

$$E \to \mathrm{e}^{2s}E, \quad \mathbf{\Phi} \to \mathrm{e}^{s}\mathbf{\Phi}, \quad z \text{ unchanged}. \tag{32}$$

Here $g, s, \mathbf{m}, \mathbf{e}$ are real scalar and vector group parameters. The remaining elements of the symmetry group include $2p+1$ Harrison–Ehlers transformations, which can be obtained by applying the above discrete maps to (29–32). Namely, applying (17) to the electric gauge (31), one gets an electric Harrison transformation (the corresponding set of parameters will be denoted as $\mathbf{h}_e$). Acting by (28) on the magnetic gauge (30) and gravitational gauge (29) one obtains a magnetic Harrison ($\mathbf{h}_m$) and Ehlers (c_E) transformations. The full group is closed by the $SL(2,R)$ S–duality (19) expressed in terms of the target space variables. This three–parametric set can be obtained by applying (17) to gravitational gauge (29), scale (32) and Ehlers transformation.

In the particular case $p = 1$, due to the local isomorphism $SO(2,3) \sim Sp(4,R)$, there exists a simple matrix generalisation of the Ernst potential [18]. Let us form the (2×2) symmetric complex matrix collecting Kähler coordinates in the following way

$$\mathcal{E} = \begin{pmatrix} E & \Phi \\ \Phi & -z \end{pmatrix}. \tag{33}$$

One can easily check that the target space metric is reproduced via

$$dl^2 = -2\mathrm{Tr}\left\{ d\mathcal{E}\left(\bar{\mathcal{E}} - \mathcal{E}\right)^{-1} d\bar{\mathcal{E}}\left(\bar{\mathcal{E}} - \mathcal{E}\right)^{-1} \right\}, \tag{34}$$

which is a direct matrix analogue of (4). Also, the three–dimensional Einstein equations take a form similar to (7):

$$\mathcal{R}_{ij} = -2\mathrm{Tr}\left\{ \left(\bar{\mathcal{E}} - \mathcal{E}\right)^{-1} \left(\partial_{(i}\mathcal{E}\right) \left(\bar{\mathcal{E}} - \mathcal{E}\right)^{-1} \partial_{j)}\bar{\mathcal{E}} \right\}. \tag{35}$$

The analogy to vacuum General Relativity is suggestive of expressing U–duality transformations in a way similar to (9) – (11) with matrix valued parameters. The gauge transformation (9) now is uplifted to

$$\mathcal{E} \to \mathcal{E} + B, \tag{36}$$

where B is the real matrix of parameters

$$B = \begin{pmatrix} g & m \\ m & b \end{pmatrix}. \tag{37}$$

This matrix–valued gauge transformation joins a gravitational gauge (g), magnetic gauge (m) and an axion shift (b) belonging to S–duality (cf. (9)).

The scale transformation (10) now is split into a symmetry–preserving matrix relation:

$$\mathcal{E} \to A^T \mathcal{E} A. \tag{38}$$

Apart from the genuine $SL(2, R)$ scale (a), it includes gravitational scale (s), electric gauge (e) and electric Harrison (h_e) transformations:

$$A = \begin{pmatrix} \mathrm{e}^s & h_e \\ -e & a \end{pmatrix}. \tag{39}$$

The last subgroup is the linear shift of an inverted matrix

$$\mathcal{E}^{-1} \to \mathcal{E}^{-1} + C, \tag{40}$$

where C is a real symmetric matrix of parameters

$$C = \begin{pmatrix} c_E & h_m \\ h_m & c \end{pmatrix}, \tag{41}$$

combining c–transformation of S–duality with magnetic Harrison (h_m) and Ehlers (c_E) transformations. For pure dilaton–axion gravity without vector fields the matrices B, A, C become diagonal and correspond to the product of two $SL(2, R)$ factors. Dilaton–axion gravity with one vector field generates $Sp(4, R)$ symmetry, as was first found in [22]. Now to make contact with the $Sp(4, R)$ group, one has merely to decompose the matrix Ernst potential into two symmetric real matrices [18]

$$\mathcal{E} = \mathcal{Q} + i\mathcal{P}, \tag{42}$$

and then construct a 4×4 real matrix

$$\mathcal{M} = \begin{pmatrix} \mathcal{P}^{-1} & \mathcal{P}^{-1}\mathcal{Q} \\ \mathcal{Q}\mathcal{P}^{-1} & \mathcal{P} + \mathcal{Q}\mathcal{P}^{-1}\mathcal{Q} \end{pmatrix}. \tag{43}$$

This is a symmetric symplectic matrix satisfying

$$\mathcal{M}^T J \mathcal{M} = J, \quad J = \begin{pmatrix} O & I_2 \\ -I_2 & O \end{pmatrix}. \tag{44}$$

In terms of $\mathcal{M}$ the metric of the target space reads

$$dl^2 = -\frac{1}{4}\mathrm{Tr}\{d\mathcal{M} d\mathcal{M}^{-1}\}, \tag{45}$$

while the Einstein equations for h_{ij} are

$$\mathcal{R}_{ij} = -\frac{1}{4}\mathrm{Tr}\{\left(\partial_{(i}\mathcal{M}\right)\partial_{j)}\mathcal{M}^{-1}\}. \tag{46}$$

A similar decomposition of U–duality can be constructed for arbitrary p, but the associated matrix structures are more involved, so we will not pursue this here. Rather, let us consider one other manifestation of the "square" property of dilaton–axion gravity related to further dimensional reduction. If, in addition to stationarity, an assumption of axial symmetry is made (more generally, that of existence of two commuting spacetime Killing vectors), the rotation one form in (1) will have its only non–zero component $\omega_\varphi = \omega$ corresponding to rotation along the symmetry axis, while the three–metric can be written in the Lewis–Papapetrou gauge

$$h_{ij}dx^i dx^j = \mathrm{e}^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2. \tag{47}$$

Then γ disappears from the dynamical equations for the σ–model variables which now take the form of a modified chiral matrix

$$\left(\rho\mathcal{M}_{,\rho}\mathcal{M}^{-1}\right)_{,\rho} + \left(\rho\mathcal{M}_{,z}\mathcal{M}^{-1}\right)_{,z} = 0. \tag{48}$$

This can serve as a standard input for an application of of integrable systems techniques [9, 12]. A Lax representation can be written straightforwardly in terms of $\mathcal{M}$. Vacuum Einstein theory is a particular case of this system with $\phi = \kappa = v = u = 0$. In that case there exists an alternative chiral equation involving another matrix $\mathcal{F}$ which is expressed directly through f and ω. Since ω and the NUT potential are related non–locally via (2), $\mathcal{M}$ and $\mathcal{F}$ representations are essentially different. Meanwhile a point–like relation between two pairs f, χ and f, ω exists, known as the Kramer–Neugebauer (KN) map, which transforms $\mathcal{M}$–equations into $\mathcal{F}$–equations and vice–versa. This map is particularly helpful in obtaining the elements of the Geroch group explicitly. The $\mathcal{F}$–representation for dilaton–axion gravity was found in [23]:

$$\mathcal{F} = \begin{pmatrix} \mathcal{P} & -\mathcal{P}\Omega \\ -\Omega\mathcal{P} & \Omega\mathcal{P}\Omega - \rho^2\mathcal{P}^{-1} \end{pmatrix}. \tag{49}$$

Here Ω is a real symmetric matrix

$$\Omega = \begin{pmatrix} \omega & -q \\ -q & qv - \beta \end{pmatrix}, \qquad q = a + v\omega, \tag{50}$$

$a = A_\varphi$ is the spatial component of the vector potential, and $\beta = B_{0\varphi}$ is the component of the Kalb–Ramond field (which was at the origin of κ). The matrix Ω generalises the ω of General Relativity, $\mathcal{P}$ replaces the scalar f, while the quantity $\mathcal{Q}$ which enters (43) is a matrix analogue of (minus) χ. Similarly to the two–dimensional reduction of (2), there exists a non–local relation between $\mathcal{Q}$ and Ω :

$$\nabla\mathcal{Q} = -\rho^{-1}\mathcal{P}\left(\tilde{\nabla}\Omega\right)\mathcal{P}, \tag{51}$$

where $\nabla = (\partial_\rho, \partial_z)$ and $\tilde{\nabla} = (\partial_z, -\partial_\rho)$ are two–dimensional Hodge dual operators. Now, the *local* map between $\mathcal{M}$ and $\mathcal{F}$ is realised by

$$\mathcal{Q} \to i\Omega, \qquad \mathcal{P} \to \rho\mathcal{P}^{-1}. \tag{52}$$

To see this it is sufficient to write down equations for the ($\mathcal{P}$, $\mathcal{Q}$) and ($\mathcal{P}$, Ω) pairs following from equations (48) for $\mathcal{M}$ and $\mathcal{F}$ [23]. The relation (52) is a direct "matrix square" of the original KN map $\chi \to -i\omega$, $f \to \rho/f$. Note, that in the both cases i does not imply complexification, but is needed just to accommodate the different signature of cosets relevant to two alternative representations. Similar a KN map exists for arbitrary number p of vector fields. For $p = 0$ it was given earlier in [24]. (The application of the integrable systems techniques to this case was recently discussed by Bakas [25].)

To summarise: coupling of the dilaton–axion system to gravity leads to a three–dimensional σ–model with a Kähler target manifold being a "square" of the corresponding General Relativity manifold. When vector fields are added, the Kähler potential still preserves its value given by the product of Ernst and axidilaton Kähler potentials. This gives rise to various manifestations of the square property of the $N = 4$ supergravity with respect to General Relativity and provides new tools in the search of classical solutions to this theory.

The author wishes to thank the Department of Physics and Mathematical Physics of the Adelaide University for the invitation to participate in the First Australasian Conference on General Relativity and Gravitation, and for hospitality and support during his visit. Stimulating discussions with P. Szekeres, D. Wiltshire, S. Scott, E. Fackerell and C. Cosgrove are gratefully acknowledged. The final version of the paper was written while the author was visiting the University of Campinas supported by FAPESP, where P. Letelier provided a stimulating environment and discussions. This work was supported in part by the RFBR Grant 96–02–18899.

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