Quantum cosmology, supersymmetry, and the problem of time

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Abstract

Quantum cosmology is the application of quantum theory to the Universe as a whole, including the gravitational degrees of freedom. It has been put forward as a way of understanding various features of the Universe (such as its isotropy), and also offers an interesting picture of its origin. However, there remain several fundamental questions about the implementation and interpretation of this approach. In particular, there is much uncertainty about the role of time in quantum cosmology and the interpretation of the wave function of the Universe.

Recent work has raised hopes that supersymmetry might shed light on these questions. Indeed, supersymmetry simplifies the mathematical structure of the quantum theory and suggests entirely new approaches to some of the most important outstanding issues.

There follows a brief review of the ideas underlying quantum cosmology, and of the role which might be played by supersymmetry. There is also a short summary of work done recently with Robert Graham on the emergence of a cosmological time parameter in quantum supergravity, which provides a possible solution to the "Problem of Time".

1. Introduction

Any attempt to describe the entire Universe as a quantum system must address the problem of quantising gravity. Unfortunately, this question is bedevilled by profound conceptual and technical problems. The fundamental nature of these problems is a reflection of our inexperience with quantum phenomena outside the rather restrictive context of quantum electrodynamics. It is hardly surprising that a paradigm based on a relatively narrow range of physical phenomena should become rather murky when we try to extend it to events such as the origin of the Universe itself.

However, there is nothing to stop us trying! And indeed, a number of plausible approaches have been suggested. In this section, I will briefly discuss a few of these approaches and their relative strengths and weaknesses.

Perhaps conceptually simplest is the functional integral approach, in which one aims to calculate the quantum transition amplitudes between two specified spatial 3-geometries by summing over all interpolating spacetime histories. (See [1] for a recent account of the technical issues and an extensive bibliography.) As well as its conceptual elegance, this approach is manifestly covariant and can easily accommodate spatial topology changes. However, when it comes to implementation, a number of technical difficulties and limitations arise. In particular, one is generally restricted to semi-classical *perturbative* calculations, which take into account only small fluctuations about classical spacetimes. While this limitation may eventually be overcome, at present it must be seen as a serious shortcoming.

On the other hand, the canonical approach to quantum gravity is genuinely non-perturbative. In this approach, one starts by considering the classical dynamics of a spatial 3-geometry evolving in time and then replaces the dynamical variables by operators acting on the wave function. The procedure is clearly not covariant, although one hopes that covariance will be recovered when one considers physically observable quantities.

Another criticism of canonical approach to quantum gravity is that it does not readily accommodate the possibility of spatial topology changes. While this suggests that the canonical approach may not provide an ultimate and all-encompassing description, it does not detract from the usefulness of the canonical approach when considering problems in which the spatial topology is fixed.

There are several quite different approaches to canonical quantum gravity, reflecting the different ways in which one can choose the dynamical variables describing the 3-geometry. The first of these goes back to Wheeler and Dewitt, who chose the canonical coordinates as the 3-metric [2, 3]. This approach has subsequenly been investigated by many authors, and has given rise to many technical questions which are still unanswered [4]. In particular, it has not been clear how one should define the inner product on the Hilbert space of states, which is needed in order to understand the physical significance of the wave function. A related issue concerns the ambiguity in the choice of the time variable [5]. These problems, and a possible solution, will be discussed in the subsequent sections.

A rather different approach to canonical quantum gravity has also been developed by Ashtekar and his followers, in which the canonical coordinates describing the 3-geometry are chosen as the self-dual part of the connection form rather than the metric tensor. While this approach also runs into certain technical difficulties, it nonetheless appears very promising. Recently there has also been considerable interest in a related idea by Rovelli and Smolin [6], who proposed that the fundamental dynamical variables might be chosen as the holonomies of closed loops. For an account of both these approaches, the reader is referred to Ashtekar's book [7].

2. The Canonical Formulation of Classical General Relativity

Our review of classical General Relativity starts with the Einstein-Hilbert action

$$S = \int_{\mathcal{M}} (R + \mathcal{L}_{matter}) \sqrt{-g} d^4x \tag{1}$$

where R is the Ricci curvature scalar obtained from the spacetime metric $g_{\mu\nu}$, $\mathcal{L}_{matter}$ is the Lagrangian density for the matter fields and $\sqrt{-g}\, d^4x$ is the invariant 4-volume[8].

Requiring this action to be stationary with respect to $g_{\mu\nu}$ leads directly to Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}. \tag{2}$$

To proceed to the Hamiltonian formulation of the theory, it is convenient to think of spacetime as the history of an evolving spatial 3-manifold $\Sigma(t)$. At each time t, $\Sigma(t)$ is parameterised by local coordinates x^i, $i = 1, 2, 3$ and is equipped with a spacelike 3-metric h_{ij} inherited from the spacetime metric, which can now be written in the form

$$ds^2 = (N^i N_i - N^2)dt^2 \ + \ 2N_i dt\, dx^i \ + \ h_{ij}dx^i dx^j. \tag{3}$$

The variable N is called the Lapse function, since Ndt is the proper time interval between two hypersurfaces separated by a coordinate time difference dt. We also have $N_i = h_{ij}N^j$ where the 3-vector N^j is referred to as the Shift vector. ($N^i dt$ is essentially the coordinate difference between points on two such hypersurfaces which are joined by normal geodesics[3, 8].)

For the sake of simplicity let us suppose that no matter is present, so that $\mathcal{L}_{matter} = 0$. The Einstein-Hilbert action can then be written

$$S = \int L\, dt \tag{4}$$

where the Lagrangian L at time t is given by

$$L = \int_{\Sigma(t)} h^{1/2} d^3x\, N(K^{ij}K_{ij} - K^2 + {}^{(3)}R). \tag{5}$$

Here, ${}^{(3)}R$ denotes the Ricci scalar curvature derived from the 3-metric h_{ij} intrinsic to the hypersurface $\Sigma(t)$, while $h \equiv \det[h_{ij}]$, and

$$K_{ij} \equiv \frac{1}{2N}\left(N_{i|j} + N_{j|i} - \frac{\partial h_{ij}}{\partial t}\right) \tag{6}$$

are the components of the extrinsic curvature of $\Sigma(t)$. The vertical bars appearing in the subscripts denote covariant differentiation within the 3-dimensional space $\Sigma(t)$.

The momenta conjugate to the dynamical variables h_{ij} at each point in $\Sigma(t)$ are now found to be

$$\Pi^{ij} = \frac{\delta L}{\delta \dot{h}_{ij}} = h^{\frac{1}{2}}[K^{kl}h_{kl}h^{ij} - K^{ij}] \tag{7}$$

where $\dot{h}_{ij} \equiv \partial h_{ij}/\partial t$. Similarly, the momenta associated with the dynamical variables N and N_i at each point in $\Sigma(t)$ are

$$\Pi = \frac{\delta L}{\delta \dot{N}} = 0, \qquad \Pi^i = \frac{\delta L}{\delta \dot{N}_i} = 0 \tag{8}$$

where $\dot{N} \equiv \partial N/\partial t$ and $\dot{N}_i \equiv \partial N_i/\partial t$. The vanishing momenta Π and Π^i are referred to as *primary constraints* [9].

In any classical theory, the Hamiltonian is obtained from the Lagrangian by a Legendre transformation of the form $H = p_\alpha \dot{q}^\alpha - L$. In this case therefore the Hamiltonian is just

$$H = \int_{\Sigma(t)} \left(\Pi \dot{N} + \Pi^i \dot{N}_i + \Pi^{ij} \dot{h}_{ij} \right) d^3x \quad - \; L. \tag{9}$$

Eliminating the velocities $\dot{h}_{ij}$ in favour of the momenta Π^{ij} allows us to rewrite this as

$$H = \int_{\Sigma(t)} \left(\Pi \dot{N} + \Pi^i \dot{N}_i + N\mathcal{H} + N_i \mathcal{H}^i \right) d^3x \tag{10}$$

where

$$\mathcal{H} = \frac{1}{2} h^{-1/2} (h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}) \Pi^{ij}\Pi^{kl} - h^{1/2\,(3)}R \tag{11}$$

and

$$\mathcal{H}^i = -2\Pi^{ij}{}_{|j}. \tag{12}$$

(We assume here that the 3-manifold $\Sigma(t)$ is compact and without boundary. If $\Sigma(t)$ had a boundary, then the expression for the Hamiltonian H would also include spatial boundary terms.)

It is clear from (8) that the velocities $\dot{N}$ and $\dot{N}_i$ cannot be expressed in terms of the momenta Π and Π^i. Nonetheless, the constraints can still be exploited to eliminate $\dot{N}$ and $\dot{N}_i$ from the expression for the Hamiltonian so that the latter is a function only of canonical coordinates and momenta.

Indeed, because of the constraints (8), one is free to add any multiples of Π and Π_i to the Hamiltonian without affecting its value. Thus, we are free to redefine the Hamiltonian as

$$H = \int_{\Sigma(t)} \left(\lambda \Pi + \lambda_i \Pi^i + N\mathcal{H} + N_i \mathcal{H}^i \right) d^3x \tag{13}$$

where λ and λ_i at each spacetime point are arbitrary quantities, which may be taken as any prespecified functions of the coordinates (x^i, t), the dynamical variables (N, N_i, h_{ij}, Π, Π^i, Π^{ij}) and their derivatives. (It is easily checked that such a redefinition of H will not affect the validity of Hamilton's equations of motion.) The quantities λ and λ_i may be viewed as Lagrange multipliers, since requiring the Hamiltonian (13) to be stationary with respect to their variation leads directly to the primary constraints (8).

The Hamiltonian (13) is the generator of time translations, as it should be. Indeed, the rate of change of any functional $F[N, N_i, h_{ij}; \Pi, \Pi^i, \Pi^{ij}]$ is found to be

$$\frac{dF}{dt} = \{f, H\} \tag{14}$$

where $\{,\}$ denotes the (functional) Poisson bracket;

$$\begin{aligned} \{F, G\} \equiv \int_\Sigma d^3x \Big[& \frac{\delta F}{\delta N(\mathbf{x})} \frac{\delta G}{\delta \Pi(\mathbf{x})} + \frac{\delta F}{\delta N_i(\mathbf{x})} \frac{\delta G}{\delta \Pi^i(\mathbf{x})} + \frac{\delta F}{\delta h_{ij}(\mathbf{x})} \frac{\delta G}{\delta \Pi^{ij}(\mathbf{x})} \\ & - \frac{\delta F}{\delta \Pi(\mathbf{x})} \frac{\delta G}{\delta N(\mathbf{x})} - \frac{\delta F}{\delta \Pi^i(\mathbf{x})} \frac{\delta G}{\delta N_i(\mathbf{x})} - \frac{\delta F}{\delta \Pi^{ij}(\mathbf{x})} \frac{\delta G}{\delta h_{ij}(\mathbf{x})} \Big] \end{aligned} \tag{15}$$

In particular, at each point $\mathbf{x}$ in $\Sigma(t)$ one has

$$\dot{N}(\mathbf{x}) = \{N(\mathbf{x}), H\} = \lambda(\mathbf{x}), \qquad \dot{N}_i(\mathbf{x}) = \{N_i(\mathbf{x}), H\} = \lambda_i(\mathbf{x}). \tag{16}$$

If we wish, we can use these identities to eliminate λ and λ_i from the modified Hamiltonian (13); the latter will then reduce to the original form (15).

We now return to the constraints (8). Since $\Pi(\mathbf{x})$ and $\Pi^i(\mathbf{x})$ vanish at all times, then so must their time derivatives $\dot{\Pi}(\mathbf{x})$ and $\dot{\Pi}^i(\mathbf{x})$. In fact

$$\dot{\Pi}(\mathbf{x}) = \{\Pi(\mathbf{x}), H\} = \mathcal{H}(\mathbf{x}), \qquad \dot{\Pi}^i(\mathbf{x}) = \{\Pi^i(\mathbf{x}), H\} = \mathcal{H}^i(\mathbf{x}) \tag{17}$$

where $\mathcal{H}(\mathbf{x})$ and $\mathcal{H}^i(\mathbf{x})$ are as defined in (11, 12), and so the vanishing of $\dot{\Pi}$ and $\dot{\Pi}^i$ gives rise at each point $\mathbf{x} \in \Sigma(t)$ to the *secondary constraints*

$$\mathcal{H}(\mathbf{x}) = 0 \tag{18}$$

and

$$\mathcal{H}^i(\mathbf{x}) = 0. \tag{19}$$

Equations (18) are referred to as the "Hamiltonian constraints" for canonical general relativity, and express the invariance of the theory under time reparameterisations. On the other hand, (19) are known as the "momentum" constraints, and express the invariance of the theory under coordinate transformations within the three-dimensional manifold $\Sigma(t)$.

In fact the secondary constraints $\mathcal{H}(\mathbf{x}) = 0$ and $\mathcal{H}^i(\mathbf{x}) = 0$ are equivalent to four of the ten Einstein's vacuum field equations; namely,

$$R^{00} - \frac{1}{2}Rg^{00} = 0, \qquad R^{0i} - \frac{1}{2}Rg^{0i} = 0. \tag{20}$$

The content of the remaining six Einstein equations ($R^{ij} - \frac{1}{2}Rg^{ij} = 0$) is contained in the evolution equations for the 3-metric $h_{ij}(\mathbf{x})$ and the conjugate momenta $\Pi^{ij}(\mathbf{x})$; namely,

$$\dot{h}_{ij}(\mathbf{x}) = \{h_{ij}(\mathbf{x}), H\}, \qquad \dot{\Pi}^{ij}(\mathbf{x}) = \{\Pi^{ij}(\mathbf{x}), H\}. \tag{21}$$

Note that the last two equations are special cases of the general evolution equation (14).

In concluding this section, it must be emphasised that the evolution generated by equations (14) will inevitably depend how we choose the quantities λ and λ_i. That is, we are forced to make a definite choice of λ and λ_i to get a well-defined classical theory with unambiguous evolution equations. (Choosing these functions is analogous to choosing coordinate conditions before solving Einstein's equations.) In other words, the *a priori* specification of λ and λ_i is an essential ingredient in the canonical formulation of the classical theory.

3. Canonical Quantisation of General Relativity

Dirac's quantisation procedure can now be applied [9]. This means, firstly, representing the dynamical variables by operators which act on a wave functional $\Psi(t; N, N_i, h_{ij}]$. Here we use the coordinate representation, in which the canonical coordinates $N(\mathbf{x})$, $N_i(\mathbf{x})$ and $h_{ij}(\mathbf{x})$ are represented by multiplicative operators, while their respective conjugate momenta $\Pi(\mathbf{x})$, $\Pi^i(\mathbf{x})$ and $\Pi^{ij}(\mathbf{x})$ are represented by the (functional) differential operators

$$\widehat{\Pi}(\mathbf{x}) = -i\hbar\frac{\delta}{\delta N(\mathbf{x})}, \qquad \widehat{\Pi}^i(\mathbf{x}) = -i\hbar\frac{\delta}{\delta N_i(\mathbf{x})}, \qquad \widehat{\Pi}^{ij}(\mathbf{x}) = -i\hbar\frac{\delta}{\delta h_{ij}(\mathbf{x})}. \tag{22}$$

Functions of these variables, such as H, $\mathcal{H}(\mathbf{x})$ and $\mathcal{H}_i(\mathbf{x})$, are also converted to operators.

The constraints of the classical theory are converted into conditions on the wave function. The primary constraints (8) give

$$\widehat{\Pi}(\mathbf{x})\Psi = 0, \qquad \widehat{\Pi}^i(\mathbf{x})\Psi = 0 \tag{23}$$

which imply (on account of (22)) that the wave function Ψ will be independent of the variables $N(\mathbf{x})$ and $N_i(\mathbf{x})$.

On the other hand, the secondary constraints govern the dependence of Ψ on the metric h_{ij}. At each point $\mathbf{x} \in \Sigma(t)$ the Hamiltonian constraint (18) yields

$$\widehat{\mathcal{H}}(\mathbf{x})\Psi = 0 \tag{24}$$

where $\widehat{\mathcal{H}}(\mathbf{x})$ is a hyperbolic differential operator which is second-order in $\delta/\delta h_{ij}(\mathbf{x})$. ($\widehat{\mathcal{H}}(\mathbf{x})$ is obtained by substituting $-i\hbar\delta/\delta h_{ij}(\mathbf{x})$ for $\Pi^{ij}(\mathbf{x})$ in (11) after choosing some factor-ordering). This is known as the "Wheeler-DeWitt" equation. Similarly, the momentum constraints (19) give

$$\widehat{\mathcal{H}^i}(\mathbf{x})\Psi = 0 \tag{25}$$

at each point $\mathbf{x}$, where the differential operator $\widehat{\mathcal{H}_i}(\mathbf{x})$ is first-order in $\delta/\delta h_{ij}(\mathbf{x})$. These conditions ensure that the wave function Ψ is unaffected by coordinate transformations within the 3-manifold $\Sigma(t)$; i.e. that Ψ depends only on the *geometry* of $\Sigma(t)$.

The third step in the quantisation procedure is to impose on $\Psi(t; h_{ij}]$ the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\Psi, \tag{26}$$

which in this case reduces to

$$i\hbar\frac{\partial\Psi}{\partial t} = \int_{\Sigma(t)} \left(\lambda\widehat{\Pi} + \lambda_i\widehat{\Pi}^i + N\widehat{\mathcal{H}} + N_i\widehat{\mathcal{H}^i}\right) d^3x\Psi = 0 \tag{27}$$

on account of the form of the Hamiltonian (13) and the conditions (23), (24)and (25). We therefore have a *time-independent* wave function $\Psi[h_{ij}]$.

The time independence of the wave function $\Psi[h_{ij}]$ seems strange, until one remembers that t is nothing more than an arbitrary labelling system for the collection of 3-manifolds comprising spacetime. The parameter t has no physical meaning, and so in retrospect it is not surprising that the Schrödinger equation tells us nothing about what happens to Ψ when t is varied.

Of course, the absence of any time parameter in the theory is hard to reconcile with our everyday experience; indeed, this represents one of the most puzzling and profound problems in quantum gravity and quantum cosmology. There are a variety of approaches. For example, taking an operational view of time as a physical observable, one might postulate the existence of a operator whose eigenvalues correspond to the time shown on a particular clock. However, one is then faced with the difficulty of deciding which clock should be used; or else, of showing that this choice does not in fact matter [4, 5].

Closely related to the problem of recovering time from the timeless quantum theory predicted by (27), is the problem of interpreting the wave function Ψ. In conventional quantum theory, we know that the integral (over some region in the configuration space) of $|\Psi|^2$ can be interpreted as a probability. This interpretation makes sense for 3 reasons;

- the integral of $|\Psi|^2$ is real and non-negative, as required for a probability
- the integral of $|\Psi|^2$ over the whole configuration space is conserved with respect to time; hence, the total probability is conserved.
- this interpretation can be been experimentally confirmed by making numerous measurements of identical systems in a specified quantum state.

Unfortunately, none of these arguments can be applied to the cosmological wave function Ψ. In the first place, it is not clear how to evaluate the integral of $|\Psi|^2$ over a region in configuration space, as we do not know what measure should be used. In the second place, even if we agree on a measure, we cannot claim that the integral of $|\Psi|^2$ is conserved with respect to time, since the theory is time independent. Thirdly, we can never experimentally confirm such an interpretation, as we cannot make numerous independent measurements on an ensemble of Universes in the same quantum state.

It is therefore not at all clear how we should interpret the cosmological wave function Ψ, and what it tells us about the Universe. Without such an interpretation, we cannot reasonably claim to have a meaningful theory.

Another fundamental problem is how to determine which quantum state the Universe is actually in. This cannot be decided just by looking at the quantum constraints, which admit an uncountable infinity of different solutions. To answer this question we need to find compelling physical reasons to impose some kind of boundary condition on the wave functional Ψ.

Hartle and Hawking offered a very plausible solution by proposing that the Universe should be in its quantum mechanical ground state, defined so that the wave function gives the quantum amplitude for the spontaneous appearance of a given 3-geometry from nothing at all [10]. Such an amplitude can be obtained by adding contributions from all compact Riemannian 4-geometries with no boundary other than the specified 3-geometry; for this reason, it is called the Hartle-Hawking "no-boundary" state. The idea is extremely compelling on aesthetic (and perhaps even philosophical) grounds, though in recent years it has lost some of its initial popularity owing to unresolved technical problems. A rival proposal by Vilenkin has also attracted wide interest [11].

To conclude this section, it is useful to illustrate the mathematical structure of the constraints by studying the quantisation of spatially homogeneous cosmologies. (This is the *mini-superspace* approach to quantum cosmology [3, 12].) For example, in this case of the Bianchi IX model[8, 13], the spatial geometry can be parameterised by an overall scale factor $e^{2\alpha}$ and two anisotropy parameters β_+ and β_-. The wave function $\Psi(\alpha, \beta_+, \beta_-)$ must then satisfy a single Wheeler-DeWitt equation $\widehat{\mathcal{H}}\Psi = 0$ which follows from the classical constraint $\mathcal{H} = 0$; in the coordinate representation, this equation has the form

$$0 = \frac{\hbar^2}{2}\left[\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2}\right]\Psi + U(\alpha, \beta_+, \beta_-)\Psi \tag{28}$$

where, with a particular choice of factor-ordering, the potential U is given by

$$U(\alpha, \beta_+, \beta_-) = \frac{1}{6}e^{4\alpha}\left[2e^{4\beta_+}[\cosh(\sqrt{48}\beta_-) - 1] + e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(\sqrt{12}\beta_-)\right]. \tag{29}$$

Clearly, (28) does not look much like the Schrödinger equation. This makes it problematic to interpret Ψ in the conventional manner.

Note that the Wheeler-DeWitt equation (28) is hyperbolic, with α playing the part of a time-like coordinate. The same thing happens in the full non-homogeneous theory; the Wheeler-DeWitt equation at each point in space has the form of a hyperbolic second order equation, with the local scale factor acting as a timelike coordinate.

In the full quantum theory, the momentum constraints $\widehat{\mathcal{H}^i}\Psi = 0$ ensure that the wave function is unaffected by spatial diffeomorphisms; that is, by redefinitions of the spatial coordinates x^i. However, these constraints are absent in the reduced homogeneous model parameterised by $(\alpha, \beta_+, \beta_-)$ as there is no remaining freedom in the choice of coordinates. Consequently, the only requirement on the wave function Ψ is that it should satisfy the Wheeler-DeWitt equation (28).

However, we are still a long way from a meaningful quantum mechanical description of a Bianchi IX Universe! It is not at all clear from equation (28) what Ψ actually means, and there does not appear to be any role for a time variable. Moreover, we have yet to determine which of the infinitely many solutions of (28) would be suitable for describing a Universe such as ours.

4. Canonically Quantised Supergravity

The problems outlined above have been debated for decades, without any conclusive solutions being found. However, in the last few years, new life has been breathed into the subject by the observation that mathematical structure of the quantum theory can be simplified by the introduction of supersymmetry into the model [14, 15, 16, 17, 18]. This simplification occurs because supersymmetry is more fundamental than invariance under time reparameterisation, in the sense that the generator of time translations can be expressed as the anticommutator of supersymmetry generators. It is therefore natural to hope that supersymmetry might should shed new light on some of the old problems.

Roughly speaking, a supersymmetric theory is one with a symmetry group which includes spacetime diffeomorphisms and transformations which mix bosonic and fermionic degrees of freedom. The elegance of supersymmetry lies in the way in which it ties the bose-fermi symmetries together with the spacetime symmetries.

A theory incorporating both supersymmetry and general relativity is referred to as a *supergravity* model. Such models were studied intensively during the late seventies and early eighties[19, 20] in the hope that one could be made to accommodate all the known fundamental interactions within a mathematically consistent framework. Although no such model was found, it is still widely expected that supersymmetry in some form or other (such as superstrings) should play a role in any satisfactory Theory of Everything. (This is certainly my point of view.) This provides an important motivation for investigating the consequences of supersymmetry in quantum cosmology.

In the absence of any more realistic supersymmetric theory incorporating General Relativity, it is natural to start by considering the canonical quantisation of the simplest such model; pure $N = 1$ supergravity [20]. A thorough analysis of the canonical theory was given by D'Eath in 1984 [21], and a very brief summary of this work is given below.

The most important manifestation of supersymmetry in the canonical theory is the appearance of 10 new constraints at each point $\mathbf{x}$, in addition to those which arise in ordinary general relativity. Six of these are the angular momentum constraints, which have the form

$$J_{ab}(\mathbf{x}) = 0 \tag{30}$$

where $a, b = 0, 1, 2, 3$ are Lorentz indices and $J_{ab}(\mathbf{x}) = J_{[ab]}(\mathbf{x})$ are the generators of local Lorentz rotations. These simply express the Lorentz invariance of the theory. The other four new constraints reflect the invariance of the theory under supersymmetry transformations, and have the form

$$S_A(\mathbf{x}) = 0, \qquad \bar{S}_{A'}(\mathbf{x}) = 0 \tag{31}$$

where $A, A' = 0, 1$ are two-component spinor indices, and $S_A(\mathbf{x})$, $\bar{S}_{A'}(\mathbf{x})$ are the generators of local supersymmetry transformations.

The algebraic structure of the canonical theory is described by listing the Dirac brackets between the various constraints [21]. (The "Dirac bracket" is a generalisation of the familiar Poisson bracket, having essentially the same properties but taking account of the fact that the fermion fields and their conjugate momenta are not independent [9, 22].) One finds that the Dirac bracket of any two constraints in canonical supergravity is itself a linear combination of constraints; hence the set of constraints closes under action of the Dirac bracket and forms what is called the "Dirac algebra" of the theory.

Of particular interest is the Dirac bracket of the supersymmetry constraints $S_A(\mathbf{x})$ and $S_{A'}(\mathbf{x})$, which turns out to be a linear combination of the Lorentz constraints $J_{ab}(\mathbf{x})$ and the Hamiltonian and momentum constraints $\mathcal{H}(\mathbf{x})$ and $\mathcal{H}^i(\mathbf{x})$. This relation is the origin of the intriguing interpretation of supergravity as the square root of general relativity.

The importance of the Dirac bracket relations become clear when one goes to the quantum theory, in which the Dirac bracket of two variables is replaced by the commutator or anticommutator of the corresponding operators (depending on whether the variables are bosonic or fermionic). In particular, the Dirac bracket relation between S_A and $S_{A'}$ is replaced by the anticommutation relation

$$\widehat{S}_A \widehat{\bar{S}}_{A'} + \widehat{\bar{S}}_{A'} \widehat{S}_A = \text{linear combination of } \widehat{J}_{ab},\ \widehat{\mathcal{H}} \text{ and } \widehat{\mathcal{H}_i}\ . \tag{32}$$

Hence, if the wave function Ψ satisfies the 6 Lorentz constraints

$$\widehat{J}_{ab}(\mathbf{x})\Psi = 0 \tag{33}$$

at each point $\mathbf{x}$ as well as the 4 supersymmetry constraints

$$\widehat{S}_A(\mathbf{x})\Psi = 0, \qquad \widehat{\bar{S}}_{A'}(\mathbf{x})\Psi = 0, \tag{34}$$

then it will automatically satisfy the remaining constraints

$$\widehat{\mathcal{H}}(\mathbf{x})\Psi = 0\ , \qquad \widehat{\mathcal{H}_i}(\mathbf{x})\Psi \tag{35}$$

everywhere in Σ. (To see this, one simply notes that Ψ is annihilated by the operator on the left-hand side of (32) and hence also by the operator on the right-hand side, for all values of $A, A' = 0, 1$.) We are therefore able to entirely forget about the Hamiltonian and momentum constraints, provided that the wave function Ψ is Lorentz invariant and is annihilated the supersymmetry generators $\widehat{S}_A$ and $\widehat{\bar{S}}_{A'}$.

The advantage of this approach is that the operators $\widehat{S}_A$ and $\widehat{\bar{S}}_{A'}$ are much simpler in form than $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}_i}$. In particular, $\widehat{\mathcal{H}}$ is second-order whereas $\widehat{S}_A$ and $\widehat{\bar{S}}_{A'}$ have the relatively simple first-order form (ignoring all indices)

$$\widehat{S} = \bar{\psi}\cdot\frac{\delta}{\delta e} + (\nabla\times\bar{\psi}) \qquad \widehat{\bar{S}} = \psi\cdot\frac{\delta}{\delta e} - (\nabla\times\psi) \tag{36}$$

where e is short for $e^a{}_i(\mathbf{x})$ and denotes the spatial part of the tetrad (defined so that $\eta_{ab}e^a{}_i e^b{}_j = h_{ij}$), while $\bar{\psi}$ and ψ respectively are abbreviations for the fermion creation operators $\bar{\psi}^{A'}{}_i(\mathbf{x})$ and annihilation operators $\psi^A{}_i(\mathbf{x})$.

The nature of the supersymmetry constraints can be illuminated by restricting Ψ to spatially homogeneous fields and Bianchi type-IX metrics. The independence of the fields on the spatial coordinates means that in this case there are only four supersymmetry constraints,

$$\widehat{S}_A\Psi = 0, \quad A = 0,1; \qquad \widehat{\bar{S}}_{A'}\Psi = 0, \quad A' = 0,1. \tag{37}$$

Each of these can be written in the Dirac form

$$0 = (\alpha^m\partial_m + \beta)\,\Psi \tag{38}$$

where X^m are coordinates on the 12-dimensional configuration space also parameterised by $e^a{}_i$, while α^m and β are complex 64×64 matrices. In fact these constraints can be viewed as square roots of the Wheeler-DeWitt equation (28), in precisely the same way that the conventional Dirac equation is the square root of the Klein-Gordon equation.

In fact it turns out that there are two distinct ways of imposing homogeneity on the Rarita-Schwinger fields $\psi^A{}_i, \bar{\psi}^{A'}{}_i$, and that the form of the matrix β depends on which choice we make [23]. Consequently there are two different versions of the supersymmetry constraints (37), one for each of the two permissible homogeneity conditions.

Because the constraints are matrix equations, the wave function Ψ will have 64 independent components. In fact, the different components $(\Psi_1, \Psi_2, \ldots \Psi_{64})$ can be grouped according to fermion number. The number of components with given fermion number are as follows;

Fermion number N_F:	0	1	2	3	4	5	6
No. of components of Ψ:	1	6	15	20	15	6	1

The components with a given fermion number together carry a representation of the Lorentz group, and can be labelled with N_F spinor indices $(A', B', \ldots)$ and N_F spatial indices $(i, j, \ldots)$. For example, the $N_F = 0$ component of the wave function is a scalar while the $N_F = 1$ components together make up a left-handed spinor 1-form $\Psi^{A'}{}_i$.

It is immediately apparent that components of the wave function with odd fermion number cannot satisfy the Lorentz constraints, since anything with an unsaturated spinor index cannot be Lorentz invariant. Indeed, it would appear that the only components of Ψ which *can* satisfy the Lorentz constraints are those represented by either a scalar or a pseudoscalar; i.e. the $N_F = 0$ and $N_F = 6$ components. (This argument was widely used and accepted [16, 17], but turns out to be incorrect. Nonetheless, let us follow it to its conclusion.)

Once we have settled on a choice of homogeneity condition (and so determined the form of the matrix β), it turns out that there is a single solution to the supersymmetry constraints with $N_F = 0$; there is also just one solution with $N_F = 6$. However, one of these two solutions is found to be non-normalisable. The argument above therefore suggests that, for a particular choice of homogeneity condition, there is a single normalisable solution to the Lorentz and supersymmetry constraints.

We thus obtain just two independent solutions, one for each permissible homogeneity conditon. Remarkably, it turns out [23] that one of these solutions can be identified as the wave function for the Hartle-Hawking "no-boundary" state discussed earlier [10]. The other solution [16] is the wave function for the "wormhole ground state", and gives the amplitude for the appearance of a closed Universe connected by a "wormhole" or throat to an asymptotically flat spacetime region [24]. However, the latter is not believed to be appropriate for describing the quantum mechanics of an expanding pseudo-Riemannian Universe such as ours. One is therefore drawn to the conclusion that the only cosmological quantum state permitted in homogeneous Bianchi IX supergravity is the Hartle-Hawking state.

The idea that supersymmetry might single out a special quantum state for the Universe caused considerable excitement in the early 1990's (among a fairly small group, admittedly!) However this excitement dissolved at the end of 1994, when Csordás and Graham showed that the arguments leading to this remarkable conclusion were incorrect, being based on a misunderstanding of the Lorentz constraints [25]. (The argument given above neglected the existence of an invariant tensor characterising the algebraic structure of the homogeneous spatial geometry. This tensor can be contracted with the free indices on certain components of Ψ to give Lorentz invariant solutions with fermion numbers $N_F = 2$ and $N_F = 4$.) Csordás and Graham demonstrated that, when the calculation is done correctly, the $N_F = 2$ and $N_F = 4$ sectors actually contain *infinitely* many Lorentz invariant solutions to the supersymmetry constraints. One thus returns to the old problem of being unable to decide which solution to choose!

5. Quantum Supergravity and Time

We saw in the last section that, in spite of its initial promise, supersymmetry does not appear to determine the quantum state of the Universe. It remains to see whether it can help solve the other fundamental problems in quantum cosmology; in particular, the algebraic connection between supersymmetry and time translation leaves us with the hope that it might shed light on the Problem of Time.

I will conclude with a brief outline of some work completed recently with Robert Graham [26, 27], showing how a cosmological time parameter arises naturally from the canonical quantisation of an elegant version of supergravity proposed by Ogievetsky and Sokatchev [28, 29]. We will see that the evolution of the wave function with respect to this time parameter is governed a set of Schrödinger equations, and can be interpreted in a very conventional way.

This work originated from an attempt to supersymmetrise the unimodular version of general relativity, obtained by varying the Einstein-Hilbert (1) action with the determinant of the metric fixed to be one [30, 31, 32]. Canonical quantisation of this theory shows that the wave function evolves with respect to t according to a type of Schrödinger equation [31]; however, the theory is somewhat unsatisfactory because the justification for the unimodular condition is unclear. By starting with the Ogievetsky-Sokatchev formulation of supergravity, we were able to obtain a similar result without the imposition of ad hoc conditions on the fields.

Conventional N=1 supergravity can be derived from a superfield theory by carefully imposing a variety of gauge conditions and making suitable transformations. The Lagrangian density for this theory is found to be $e\mathcal{L}$, where e denotes the determinant of the vielbein $e^a{}_\mu$, and

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}R + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar\psi_\mu\bar\sigma_\nu D_\rho\psi_\sigma - \psi_\mu\sigma_\nu D_\rho\bar\psi_\sigma) + \frac{1}{3}b^\mu b_\mu \\ &\quad -\frac{1}{3}M^*M - \lambda^*(M + \psi_a\sigma^{ab}\psi_b) - \lambda(M^* + \bar\psi_a\bar\sigma^{ab}\bar\psi_b), \qquad (39)\end{aligned}$$

λ is an arbitrary complex constant, b^μ is an auxiliary (i.e. non-propagating) vector field, and M is a complex auxiliary scalar field [20]. The (non-dynamical) field equations for M imply that $M = -3\lambda$; using this to eliminate M, the above expression reduces to

$$\mathcal{L} = \frac{1}{2}R + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\bar\psi_\mu\bar\sigma_\nu D_\rho\psi_\sigma - \psi_\mu\sigma_\nu D_\rho\bar\psi_\sigma) + \frac{1}{3}b^\mu b_\mu$$

$$+3\lambda^*\lambda - \lambda^*\psi_a\sigma^{ab}\psi_b - \lambda\bar{\psi}_a\bar{\sigma}^{ab}\bar{\psi}_b. \tag{40}$$

This gives rise to a theory in which the Rarita-Schwinger field has mass $m = |\lambda|$, with cosmological constant $\Lambda = -3m^2$.

However there is an alternative derivation of $N = 1$ supergravity from a superfield theory due to Ogievetsky and Sokatchev, which in many ways is simpler and more elegant [28, 29]. The resulting theory is almost identical to the conventional version described above, but there is one subtle difference; instead of being an independent scalar field, M is given by

$$M = \nabla_\mu M^\mu - \psi_a\sigma^{ab}\psi_b \tag{41}$$

where now the complex vector field M^μ must be independently varied. Requiring $\mathcal{L}$ to be stationary with respect to this variation leads to the field equation

$$0 = \partial_\nu(\nabla_\mu M^\mu - \psi_a\sigma^{ab}\psi_b) = \partial_\mu M \tag{42}$$

(rather than $M = -3\lambda$, as in the usual approach). Thus, M will be constant on-shell, but does not have to take any particular value.

Another consequence of the identity (41) is that the terms involving the constants λ, λ^* in expression (39) become total derivatives, and so can be neglected. Thus, λ and λ^* are completely eliminated from the Ogievetsky-Sokatchev theory. In this formulation, the Rarita-Schwinger mass m and the cosmological constant Λ are determined instead by the unspecified but constant quantity M; one has

$$m = \frac{1}{3}|M| \qquad \text{and} \qquad \Lambda = -\frac{1}{3}|M|^2. \tag{43}$$

In other words, m and Λ appear as dynamical quantities which are only constant by virtue of the field equations. This makes no difference in the classical theory, as we can give m or Λ any values we desire by imposing suitable initial conditions on M. However, the quantum theory is radically changed, as it is now possible to consider linear superpositions of states with different values of m or Λ.

To explore the consequences of this new feature, we now consider the canonical quantisation of the Ogievetsky-Sokatchev theory. If fields $\Lambda(x)$ and $\theta(x)$ are defined so that

$$M = \sqrt{-3\Lambda}e^{i\theta} \tag{44}$$

then we find that the canonical description of the classical theory includes the following first-class constraints at each point $\mathbf{x}$:

$$\begin{aligned}
h^{-\frac{1}{2}}\mathcal{H} &= -\Lambda - \frac{2}{3}h^{-\frac{1}{2}}h^{ij}e_{ja}n_b(\psi_i\sigma^{ab}S + \bar{\psi}_i\bar{\sigma}^{ab}\bar{S}) && (45)\\
\mathcal{H}_i &= -\frac{1}{3}(\psi_i S + \bar{\psi}_i\bar{S}) - \frac{1}{3}e_{ic}h^{jk}e_{ka}(\psi_i\sigma^{ac}S + \bar{\psi}_i\bar{\sigma}^{ac}\bar{S}) && (46)\\
S_A &= 2(-\Lambda/3)^{1/2}h^{\frac{1}{2}}e^{i\theta}n_a h^{ij}e_{jb}(\sigma^{ab}\psi_i)_A && (47)\\
\bar{S}_{A'} &= 2(-\Lambda/3)^{1/2}h^{\frac{1}{2}}e^{-i\theta}n_a h^{ij}e_{jb}(\bar{\sigma}^{ab}\bar{\psi}_i)_{A'} && (48)
\end{aligned}$$

$$\begin{aligned} J_{ab} &= 0 & (49)\\ \partial_i \Lambda &= 0 & (50)\\ \partial_i \theta &= 0 & (51) \end{aligned}$$

where $h = \det[h_{ij}]$ and the quantities $\mathcal{H}$, $\mathcal{H}_i$, S_A, $\bar{S}_{A'}$, and J_{ab} are defined precisely as in the conventional theory [21]. The constraints involving these quantities are just modified versions of those appearing in conventional supergravity. On the other hand, the constraints (50,51) are new; these imply that the fields Λ and θ can be viewed as functions only of the time coordinate t.

In fact, by making a canonical transformation one can show that dynamical variables $\Lambda(t)$ and $\theta(t)$ can be viewed as the momenta conjugate to the new dynamical variables

$$\begin{aligned} T(t) &= -\frac{1}{6\Lambda}\int_{\Sigma(t)} d^3x\, e(M^*M^t + MM^{t*}) & (52)\\ U(t) &= -\frac{i}{3}\int_{\Sigma(t)} d^3x\, e(M^*M^t - MM^{t*}) & (53) \end{aligned}$$

where M^t is the timelike component of the complex vector field M^μ. Hence, when we go to the quantum theory, the momentum operator $\widehat{\Lambda}$ can be written (in the T-representation) as

$$\widehat{\Lambda} = -i\hbar\frac{\partial}{\partial T} \tag{54}$$

in which case the Hamiltonian constraint (45) becomes

$$i\hbar\frac{\partial\Psi}{\partial T} = h^{-\frac{1}{2}}[\mathcal{H} + \frac{2}{3}h^{ij}e_{ja}n_b(\psi_i\sigma^{ab}S + \bar{\psi}_i\bar{\sigma}^{ab}\bar{S})]\Psi \tag{55}$$

at each spacetime point x.

Clearly this can be viewed as a type of Schrödinger equation if we are prepared to think of T as a time parameter. At first sight this seems implausible; however, on closer inspection, one finds that T does indeed behave suitably. At least, it does in a particular supersymmetry gauge in which ψ_t (the non-dynamical part of the Rarita-Schwinger field) is constrained to satisfy the gauge condition

$$\psi_t{}^A = \frac{2}{3}Nh^{mn}e_{ma}n_b\psi_n{}^B\sigma^{ab}{}_B{}^A + \frac{1}{3}N^m\psi_m{}^A + \frac{1}{3}N^m e_{mc}h^{\ell n}e_{na}\psi_\ell{}^A \tag{56}$$

In this gauge, one has the identity

$$\psi_a\sigma^{ab}\psi_b = 0 \tag{57}$$

and the field equations imply that T is a monotonically increasing function of the time coordinate t; in fact, up to an additive constant, $T(t)$ is simply the four-volume of spacetime preceding the spacelike hypersurface $\Sigma(t)$ on which the time coordinate t takes the specified value. We are therefore justified in viewing T as a bona fide time parameter; i.e. a satisfactory labelling system for a prespecified foliation of spacetime by non-intersecting spacelike hypersurfaces.

Note that it is now meaningful to talk about the conservation of probability with respect to the time parameter T. Indeed, if we choose an operator ordering so that $\widehat{\mathcal{H}}$, $\widehat{\mathcal{H}^i}$ are self-adjoint and $\widehat{S}_A$, $\widehat{\bar{S}}_{A'}$ are mutually adjoint with respect to some chosen measure on the configuration space, then the integral of $\Psi^*\Psi$ on this space is a conserved quantity thanks to (55). Moreover, $\Psi^*\Psi$ is positive and real, and so is naturally interpreted as a probability density function on the configuration space (precisely as in conventional quantum mechanics).

In summary, therefore, it appears that canonical quantisation of the Ogievetsky-Sokatchev formulation of supergravity leads naturally to a time-dependent quantum theory in which the wave function has a very conventional interpretation. While the simplicity of the argument is rather surprising, it also appears inescapable if one starts with this particular version of supergravity. Moreover, it suggests that the argument can probably be extended to much more general supersymmetric models.

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